Math 3013
Solutions to Problem Set 4

1. Determine whether the indicated subset is a subspace of the given $\mathbb{R}^n$.

(a) $W = \{[r, -r] \mid r \in \mathbb{R}\}$ in $\mathbb{R}^2$

- It suffices to show that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are in W then so is any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Set
$$\mathbf{v}_1 = [r_1, -r_1] \quad , \quad \mathbf{v}_2 = [r_2, -r_2]$$
Then
$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= c_1 [r_1, -r_1] + c_2 [r_2, -r_2] \\ &= [c_1r_1 + c_2r_2, -c_1r_1 - c_2r_2] \\ &= [(c_1r_1 + c_2r_2), -(c_1r_1 + c_2r_2)] \in W \end{aligned}$$

$\square$

(b) $W = \{[n, m] \mid \ n \text{ and } m \text{ are integers}\}$ in $\mathbb{R}^2$

- This subset is not closed under scalar multiplication for
$$[1, 1] \in W \quad \text{but} \quad \sqrt{2}\,[1, 1] = \left[\sqrt{2}, \sqrt{2}\right] \notin W$$
Since this subset is not closed under scalar multiplication it cannot be a subspace. $\square$

(c) $W = \{[x, y, z] \mid x, y, z \in \mathbb{R} \text{ and } z = 3x + 2\}$ in $\mathbb{R}^3$

- Consider two arbitrary vectors in W
$$\mathbf{v}_1 = [x_1, y_1, 3x_1 + 2] \quad , \quad \mathbf{v}_2 = [x_2, y_2, 3x_2 + 2]$$
we have
$$\mathbf{v}_1 - \mathbf{v}_2 = [x_1 - x_2, y_1 - y_2, 3(x_1 - x_2) + 0] \notin W$$
Since the difference of two vectors in W does not lie in W, W is not a subspace. $\square$

(d) $W = \{[x, y, z] \mid x, y, z \in \mathbb{R} \text{ and } z = 1,\ y = 2x\}$ in $\mathbb{R}^3$

- Consider two arbitrary vectors in W
$$\mathbf{v}_1 = [x_1, 2x_1, 1] \quad , \quad \mathbf{v}_2 = [x_2, 2x_2, 1]$$
we have
$$\mathbf{v}_1 - \mathbf{v}_2 = [x_1 - x_2, 2(x_1 - x_2), 0] \notin W$$

$\square$

(e) $W = \{[2x_1, 3x_2, 4x_3, 5x_4] \mid x_i \in \mathbb{R}\}$ in $\mathbb{R}^4$

- Consider two arbitrary vectors in $\mathbb{R}^4$
$$\mathbf{x} = [x_1, x_2, x_3, x_4] \quad , \quad \mathbf{x}' = [x'_1, x'_2, x'_3, x'_4]$$
Then the vectors
$$\mathbf{v}_1 = [2x_1, 3x_2, 4x_3, 5x_4] \quad , \quad \mathbf{v}_2 = [2x'_1, 3x'_2, 4x'_3, 5x'_4]$$

will be in W. We have

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= [2c_1x_1, 3c_1x_2, 4c_1x_3, 5c_1x_4] + [2c_1x'_1, 3c_2x'_2, 4c_2x'_3, 5c_2x'_4] \\ &= [2(c_1x_1 + c_2x'_1), 3(c_1x_2 + c_2x'_2), 4(c_1x_3 + c_2x'_3), 5(c_4x_1 + c_2x'_4)] \\ &= [2x''_1, 3x''_2, 4x''_3, 5x''_4] \end{aligned}$$

This vector belongs to W since

$$\mathbf{x}'' = [c_1x_1 + c_2x'_1, c_1x_2 + c_2x'_2, c_1x_3 + c_2x'_3, c_4x_1 + c_2x'_4] \in \mathbb{R}^4$$

Since an arbitrary linear combinations of two vectors in W also lies in W, W is a subspace. □

2. Prove that the line $y = mx$ is a subspace of $\mathbb{R}^2$. (Hint: write the line as $W = \{[x, mx] \mid x \in \mathbb{R}\}$.)

- It suffices to show that an arbitrary linear combinations of two vectors in W also lies in W. Set

$$\mathbf{v}_1 = [x_1, mx_1] \quad , \quad \mathbf{v}_2 = [x_2, mx_2]$$

Then

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= [c_1x_1 + c_2x_2, c_1mx_1 + c_2mx_2] \\ &= [(c_1x_1 + c_2x_2), m(c_1x_1 + c_2x_2)] \in W \end{aligned}$$

Hence, W is a subspace. □

3. Find a basis for the solution set of the following homogeneous linear systems.

$$\begin{aligned} 3x_1 + x_2 + x_3 &= 0 \\ 6x_1 + 2x_2 + 2x_3 &= 0 \\ -9x_1 - 3x_2 - 3x_3 &= 0 \end{aligned}$$

- This linear system corresponds to the following augmented matrices

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 6 & 2 & 2 & 0 \\ -9 & -3 & -3 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + 3R_1]{R_3 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The latter augmented matrix corresponds to

$$\begin{aligned} 3x_1 + x_2 + x_3 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Which is, effectively, one equation for three unknowns. Solving for x_1 in terms of x_2 and x_3 we obtain

$$x_1 = -\frac{1}{3}(x_2 + x_3)$$

So any vector of the form

$$\left[-\frac{1}{3}x_2 - \frac{1}{3}x_3, x_2, x_3\right] = x_2\left[-\frac{1}{3}, 1, 0\right] + x_3\left[-\frac{1}{3}, 0, 1\right]$$

will be a solution. We conclude that

$$\mathbf{e}_1 = \left[-\frac{1}{3}, 1, 0\right] \quad , \quad \mathbf{e}_2 = \left[-\frac{1}{3}, 0, 1\right]$$

will be a basis for the solution space. □

4. Give a geometric criterion for a set of two distinct nonzero vectors in $\mathbb{R}^n$ to be dependent.

- If two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent, then there must exist a solution of
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$$
with at least one of the coefficients c_1, c_2 not zero. Suppose (without loss of generality) that $c_2 \neq 0$. Then c_1 can not equal zero either (otherwise we'd have $c_2\mathbf{v}_2 = \mathbf{0}$ with neither c_2 or $\mathbf{v}_2$ zero). Then we can multiply both sides of this equation by $1/c_2$ to obtain
$$\frac{c_1}{c_2}\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0} \quad \Rightarrow \quad \mathbf{v}_2 = -\frac{c_1}{c_2}\mathbf{v}_1$$
So $\mathbf{v}_2$ must be a non-zero scalar multiple of $\mathbf{v}_1$. But then, this implies that $\mathbf{v}_2$ is either parallel (or anti-parallel) to $\mathbf{v}_1$. $\square$

5. Find a basis for the row space of the matrix
$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{bmatrix}$$

The row space of $\mathbf{A}$ is the span of the row vectors $\{[1,3,5,7], [2.0,4,2], [3,2,8,7]\}$ of $\mathbf{A}$ To find a basis for the span of these vectors we arrange them as the columns of a new matrix $\mathbf{A}'$
$$\mathbf{A}' = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \\ 5 & 4 & 8 \\ 7 & 2 & 7 \end{bmatrix}$$
which happens to be the transpose of our original matrix $\mathbf{A}$. We now row-reduce $\mathbf{A}'$.
$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \\ 5 & 4 & 8 \\ 7 & 2 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -7 \\ 0 & -6 & -7 \\ 0 & -12 & -14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{H}$$
The pivots of $\mathbf{H}$ are contained in the first two columns, therefore the first two columns of $\mathbf{A}'$ form a basis for the column space of $\mathbf{A}'$, which is indentical to row space of our original matrix $\mathbf{A}$. Thus,
$$\{[1,3,5,7], [2,0,4,2]\}$$
is a basis for the row space of $\mathbf{A}$.

6. Find a basis for the column space of the matrix
$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 1 & 7 & 2 \\ 6 & -2 & 0 \end{bmatrix}$$

- We'll apply the same technique used in Problem 3.
$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 1 & 7 & 2 \\ 6 & -2 & 0 \end{bmatrix} \rightarrow \begin{matrix} 1 & 7 & 2 \\ 5 & 2 & 1 \\ 2 & 3 & 1 \\ 6 & -2 & 0 \end{matrix} \rightarrow \begin{bmatrix} 1 & 7 & 2 \\ 0 & -33 & -9 \\ 0 & -11 & -3 \\ 0 & -44 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
The pivots in the row-echelon form of $\mathbf{A}$ are in the first two columns. Therefore, the first two columns of $\mathbf{A}$
$$\{[2,5,1,6], [3,2,7,-2]\}$$
will form a basis for the column space of $\mathbf{A}$. $\square$

7. Find a basis for the subspace spanned by the vectors $[1,2,1,1]$, $[2,1,0,-1]$, $[-1,4,3,8]$, $[0,3,2,5] \in \mathbb{R}^4$.

- First we form a 4×4 matrix $\mathbf{A}$ whose columns correspond to the above set of vectors.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 4 & 3 \\ 1 & 0 & 3 & 2 \\ 1 & -1 & 8 & 5 \end{bmatrix}$$

Now we row-reduce $\mathbf{A}$ to row-echelon form.

$$\xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -3 & 6 & 3 \\ 0 & -2 & 4 & 2 \\ 0 & -3 & 9 & 5 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_2]{\substack{R_2 \to -\frac{1}{3}R_2 \\ R_3 \to R_3 - \frac{2}{3}R_2}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivots of the final matrix (a row-echelon form of $\mathbf{A}$) are in the first three columns. Hence, the first three columns

$$\{[1,2,1,1],[2,1,0,-1],[-1,4,3,8]\}$$

of $\mathbf{A}$ will form a basis for the column space

$$ColSp(\mathbf{A}) = span\left([1,2,1,1],[2,1,0,-1],[-1,4,3,8],[0,3,2,5]\right)$$

$\square$

8. Determine whether the following sets of vectors are dependent or independent.

(a) $\{[1,3],[-2,-6]\}$ in $\mathbb{R}^2$.

- Let $\mathbf{v}_1 = [1,3]$ and $\mathbf{v}_2 = [-2,-6]$ and Note that

$$2\mathbf{v}_1 + \mathbf{v}_2 = 2[1,3] + [-2,-6] = [0,0]$$

so $c_1 = 2$ and $c_2 = 1$ is a non-trivial solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Thus, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent. $\square$

(b) $\{[1,-4,3],[3,-11,2],[1,-3,-4]\}$ in $\mathbb{R}^3$.

- Let $\mathbf{v}_1 = [1,-4,3]$, $\mathbf{v}_2 = [3,-11,2]$ and $\mathbf{v}_3 = [1,-3,-4]$. We shall look for non-trivial solutions of

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \\ &= c_1[1,-4,3] + c_2[3,-11,2] + c_3[1,-3,-4] \\ &= [c_1 + 3c_2 + c_3, -4c_1 - 11c_2 - 3c_3, 3c_1 + 2c_2 - 4c_3] \end{aligned}$$

or

$$\begin{aligned} c_1 + 3c_2 + c_3 &= 0 \\ -4c_1 - 11c_2 - 3c_3 &= 0 \\ 3c_1 + 2c_2 - 4c_3 &= 0 \end{aligned} \tag{1}$$

We thus examine the following augmented matrix

$$\mathbf{A} = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -4 & -11 & -3 & 0 \\ 3 & 2 & -4 & 0 \end{array}\right]$$

Row reducing this matrix yields

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -7 & -7 & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the original system of equations is equivalent to a system of 2 independent equations in 3 unknowns. This means there will be infinitely many (in fact, a one-parameter family of) solutions of (**??**). Hence, there are non-trivial solutions so the original set of vectors are linearly dependent. □

9. For each of the following matrices find the rank of the matrix, a basis for its row space, a basis for its column space, and a basis for its null space.

(a)

$$\mathbf{A} = \left[\begin{array}{cccc} 2 & 0 & -3 & 1 \\ 3 & 4 & 2 & 2 \end{array}\right]$$

Let us first row reduce the given matrix to row-echelon form.

$$\rightarrow \left[\begin{array}{cccc} 1 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{13}{8} & \frac{1}{8} \end{array}\right] \equiv \mathbf{A}'$$

The pivots of this matrix lie in columns 1 and 2. Therefore, first two columns of the orginal matrix $\mathbf{A}$ will form a basis for the column space of $\mathbf{A}$.

$$ColSp\left(\mathbf{A}\right) = span\left(\left[2,3\right], \left[0,4\right]\right)$$

The non-zero rows of the row reduced form $\mathbf{A}'$ of $\mathbf{A}$ will be basis for the row space of $\mathbf{A}$. Hence,

$$RowSp\left(\mathbf{A}\right) = span\left(\left[1, 0, \frac{-3}{2}, \frac{1}{2}\right], \left[0, 1, \frac{13}{8}, \frac{1}{8}\right]\right)$$

The null space of $\mathbf{A}$ is the solution set of $\mathbf{Ax} = \mathbf{0}$, which coincides with the solution set of $\mathbf{A}'\mathbf{x} = \mathbf{0}$, i.e. the solution set of

$$\begin{array}{lcl} x_1 - \frac{3}{2}x_3 + \frac{1}{2}x_4 = 0 & \Rightarrow & x_1 = \frac{3}{2}x_3 - \frac{1}{2}x_4 \\ x_2 + \frac{13}{8}x_3 + \frac{1}{8}x_4 = 0 & & x_2 = -\frac{13}{8}x_3 - \frac{1}{8}x_4 \\ & \Rightarrow & \mathbf{x} = \left[\begin{array}{l} \frac{3}{2}x_3 - \frac{1}{2}x_4 \\ -\frac{13}{8}x_3 - \frac{1}{8}x_4 \\ x_3 \\ x_4 \end{array}\right] = x_3 \left[\begin{array}{l} \frac{3}{2} \\ -\frac{13}{8} \\ 1 \\ 0 \end{array}\right] + x_4 \left[\begin{array}{l} -\frac{1}{2} \\ -\frac{1}{8} \\ 0 \\ 1 \end{array}\right] \\ & \Rightarrow & \mathbf{x} \in span\left(\left[\begin{array}{l} \frac{3}{2} \\ -\frac{13}{8} \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{l} -\frac{1}{2} \\ -\frac{1}{8} \\ 0 \\ 1 \end{array}\right]\right) \end{array}$$

so the vectors $\left\{\left[\frac{3}{2}, -\frac{13}{8}, 1, 0\right], \left[-\frac{1}{2}, -\frac{1}{8}, 0, 1\right]\right\}$ form a basis for the null space of $\mathbf{A}$. Finally,

$$rank\left(\mathbf{A}\right) = \dim\left(ColSp\left(\mathbf{A}\right)\right) = \dim\left(RowSp\left(\mathbf{A}\right)\right) = 2$$

(b)

$$\mathbf{A} = \left[\begin{array}{cccc} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{array}\right]$$

- We proceed as in Part (a). The matrix $\mathbf{A}$ is row equivalent to the following matrix in row-echelon form
$$\mathbf{A}' = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
Only third column of $\mathbf{A}'$ lacks a pivot; so the first, second and fourth columns of $\mathbf{A}$ will provide a basis for the column space of $\mathbf{A}$. The matrix $\mathbf{A}'$ has three non-zero rows and these will provide a basis for the row space of $\mathbf{A}$. Finally, the solution of $\mathbf{Ax} = 0$ will be
$$\left.\begin{array}{r} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \\ x_4 = 0 \\ 0 = 0 \end{array}\right\} \implies \mathbf{x} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$
so
$$\begin{aligned} \text{basis for } ColSp(\mathbf{A}) &= \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\} \\ \text{basis for } RowSp(\mathbf{A}) &= \{[1,0,-1,0], [0,1,1,0], [0,0,0,1]\} \\ \text{basis for } NullSp(\mathbf{A}) &= \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} \end{aligned}$$
$\square$

(c)
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

- This matrix row-reduces to
$$\mathbf{H} = \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -1 \end{bmatrix}$$
The pivots of $\mathbf{H}$ lie in the first three columns. Therefore, the first three columns of $\mathbf{A}$ will be a basis for the column space of $\mathbf{A}$. Each row of $\mathbf{H}$ is non-zero; therefore, each row of $\mathbf{H}$ will be a basis vector for the row space of $\mathbf{A}$. The solution set of $\mathbf{Ax} = \mathbf{0}$ coincides with the solution set of $\mathbf{Hx} = \mathbf{0}$, or
$$\begin{array}{l} 2x_1 + x_2 + 2x_4 = 0 \\ x_2 + 2x_3 + x_4 = 0 \\ -3x_3 - x_4 = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} x_1 = -\frac{5}{6}x_4 \\ x_2 = -\frac{1}{3}x_4 \\ x_3 = -\frac{1}{3}x_4 \end{array}$$
So every solution is a vector of the form
$$\mathbf{x} = \begin{bmatrix} -\frac{5}{6}x_4 \\ -\frac{1}{3}x_4 \\ -\frac{1}{3}x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -\frac{5}{6} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \in span\left(\begin{bmatrix} -\frac{5}{6} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right)$$
Thus, $\left\{\left[-\frac{5}{6}, -\frac{1}{3}, -\frac{1}{3}, 1\right]\right\}$ is a basis for the null space of $\mathbf{A}$. The rank of $\mathbf{A}$ is equal to the number of basis vectors for the column space of $\mathbf{A}$, which is 3. $\square$