Math 3013
Solutions to Problem Set 3

1. Find the inverses of the following matrices. If a matrix inverses exists; and express it as a product of elementary matrices.

(a)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

•

$$\begin{aligned} [\mathbf{A} \mid \mathbf{I}] &= \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \\ R_1 &\to R_1 - R_2 \quad \Rightarrow \quad \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right] = [\mathbf{I} \mid \mathbf{A}^{-1}] \end{aligned}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

A is an elementary matrix (it is the elementary matrix corresponding to replacing the first row by its sum with the second row). □

(b)

$$\begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$$

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$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 4 & 8 & 0 & 1 \end{array}\right]$$

$$R_2 \to R_2 - \frac{4}{3}R_1 \quad \Rightarrow \quad \left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 0 & 0 & -\frac{4}{3} & 1 \end{array}\right]$$

This matrix does not have an inverse (it's row-echelon form has zero entries along the diagonal). □

(c)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

•

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

If we perform the following the elementary row operations

$$\begin{aligned} RowOp_1 &: R_1 \to R_1 + R_3 \\ RowOp_2 &: R_2 \to R_2 + R_3 \\ RowOp_3 &: R_3 \to -R_3 \end{aligned}$$

We convert the augmented matrix $[\mathbf{A} \mid \mathbf{I}]$ to the form

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] = [\mathbf{I} \mid \mathbf{A}^{-1}]$$

So

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Our next task is to express $\mathbf{A}^{-1}$ as a product to elementary matrices. Let $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ be the elementary matrices corresponding, respectively, to the elementary row operations $RowOp_1$, $RowOp_2$, and $RowOp_3$:

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Since the inverse of $\mathbf{A}$ was created by applying these same row operations to the identity matrix we have

$$\begin{aligned} \mathbf{A}^{-1} &= \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{I} \\ &= \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 \end{aligned}$$

- Note also that

$$\mathbf{A} = \left(\mathbf{A}^{-1}\right)^{-1} = (\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1)^{-1} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}$$

Since the inverse of an elementary matrix is another (or perhaps the same) elementary matrix, The equation above tells us how to express $\mathbf{A}$ as a product of elementary matrices. All we have do to is figure out the inverses of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$. Now the inverse of an elementary matrix $\mathbf{E}$ is just the elementary matrix that *undoes* the row operation corresponding to $\mathbf{E}$. So

$$RowOp_1 \quad : \quad R_1 \to R_1 + R_3 \text{ is undone by } R_1 \to R_1 - R_3 \quad \Rightarrow \quad \mathbf{E}_1^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_2 \quad : \quad R_2 \to R_2 + R_3 \text{ is undone by } R_2 \to R_2 - R_3 \quad \Rightarrow \quad \mathbf{E}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_3 \quad : \quad R_3 \to -R_3 \text{ is undone by } R_3 \to -R_3 \quad \Rightarrow \quad \mathbf{E}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Hence, we can also express $\mathbf{A}$ as a product of elementary matrices

$$\mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

□

(d)

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 2 & 5 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

- We shall procede as in part (c).

$$RowOp_1 \quad : \quad R_2 \to R_2 - \frac{3}{2}R_1 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$RowOp_2 \quad : \quad R_3 \to R_3 - 2R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -1 & -3 & 2 & 1 \end{array}\right]$$

$$RowOp_3 \quad : \quad R_3 \to -R_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -1 & 3 & -2 & -1 \end{array}\right]$$

$$RowOp_4 \quad : \quad R_2 \to 2R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

$$RowOp_5 \quad : \quad R_1 \to \frac{1}{2}R_1 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

$$RowOp_6 \quad : \quad R_2 \to R_2 + 2R_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

$$RowOp_7 \quad : \quad R_1 \to R_1 - \frac{1}{2}R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

$$RowOp_8 \quad : \quad R_1 \to R_1 - 2R_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 5 & 3 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]$$

So

$$\mathbf{A}^{-1} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}$$

To express $\mathbf{A}$ as a product of elementary matrices we procede as in part (c): we write

$$\mathbf{A} = (\mathbf{E}_8\mathbf{E}_7\mathbf{E}_6\mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1)^{-1} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}\mathbf{E}_4^{-1}\mathbf{E}_5^{-1}\mathbf{E}_6^{-1}\mathbf{E}_7^{-1}\mathbf{E}_8^{-1}$$

where the $\mathbf{E}_1, \ldots, \mathbf{E}_8$ are the elementary matrices corresponding to the elementary row operations $RowOp_1, \ldots, RowOp_8$, and the $\mathbf{E}_1^{-1}, \ldots, \mathbf{E}_8^{-1}$ are their matrix inverses. We have

$$RowOp_1 \quad : \quad R_2 \to R_2 - \frac{3}{2}R_1 \quad \Rightarrow \quad \mathbf{E}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_2 \quad : \quad R_3 \to R_3 - 2R_2 \quad \Rightarrow \quad \mathbf{E}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$RowOp_3 \quad : \quad R_3 \to -R_3 \quad \Rightarrow \quad \mathbf{E}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$RowOp_4 \quad : \quad R_2 \to 2R_2 \quad \Rightarrow \quad \mathbf{E}_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_5 \quad : \quad R_1 \to \frac{1}{2}R_1 \quad \Rightarrow \quad \mathbf{E}_5^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_6 \quad : \quad R_2 \to R_2 + 2R_3 \quad \Rightarrow \quad \mathbf{E}_6^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_7 \quad : \quad R_1 \to R_1 - \frac{1}{2}R_2 \quad \Rightarrow \quad \mathbf{E}_7^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_8 \quad : \quad R_1 \to R_1 - 2R_3 \quad \Rightarrow \quad \mathbf{E}_8^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$\begin{aligned} \mathbf{A} \quad = & \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ & * \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$\square$

2. Find the inverse of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

- Setting

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

and applying the following row operations

$$\begin{aligned} R_2 &\rightarrow -R_2 \\ R_3 &\rightarrow \frac{1}{2}R_3 \\ R_4 &\rightarrow \frac{1}{3}R_4 \\ R_5 &\rightarrow \frac{1}{4}R_5 \\ R_6 &\rightarrow \frac{1}{5}R_6 \end{aligned}$$

We obtain

$$\left[\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \end{array}\right] = \left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$$

So

$$\mathbf{A}^{-1} = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \end{array}\right]$$

$\square$

3. Determine whether the span of the column vectors of the given matrix is $\mathbb{R}^4$.

$$\left[\begin{array}{cccc} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 1 & 0 & -1 & 2 \\ -3 & 0 & 0 & -1 \end{array}\right]$$

- According to a theorem in the text, all we have do is show that this matrix is row equivalent to the identity matrix. After carrying out the following two row operations

$$\begin{aligned} R_3 &\rightarrow R_3 - R_1 \\ R_4 &\rightarrow R_4 + 3R_1 \end{aligned}$$

we obtain

$$\left[\begin{array}{cccc} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 3 & -4 \end{array}\right]$$

If we now carry out

$$R_4 \rightarrow R_4 + \frac{3}{2}R_3$$

we obtain

$$\left[\begin{array}{cccc} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

This matrix is clearly row-equivalent to the identity matrix (it's in row-echelon form with no zero entries along the diagonal). Hence the span of the column vectors of the original matrix must be $\mathbb{R}^4$. $\square$

4. Show that the following matrix is invertible and find its inverse.

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 5 & -7 \end{bmatrix}$$

- Setting

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 5 & -7 & 0 & 1 \end{array}\right]$$

and carrying out row reduction

$$\begin{array}{lcll} R_2 & \to & R_2 - \frac{5}{2}R_1 & \Rightarrow \quad \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right] \\ R_2 & \to & 2R_2 & \Rightarrow \quad \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & 1 & -5 & 2 \end{array}\right] \\ R_1 & \to & R_1 + 3R_2 & \Rightarrow \quad \left[\begin{array}{cc|cc} 2 & 0 & -14 & 6 \\ 0 & 1 & -5 & 2 \end{array}\right] \\ R_1 & \to & \frac{1}{2}R_1 & \Rightarrow \quad \left[\begin{array}{cc|cc} 1 & 0 & -7 & 3 \\ 0 & 1 & -5 & 2 \end{array}\right] \end{array}$$

So

$$\mathbf{A}^{-1} = \begin{bmatrix} -7 & 3 \\ -5 & 2 \end{bmatrix}$$

$\square$

5. Let

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

If possible, find a matrix $\mathbf{C}$ such that

$$\mathbf{AC} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}$$

- Multiplying (from the left) both sides of the second equation by $\mathbf{A}^{-1}$ yields

$$\mathbf{A}^{-1}(\mathbf{AC}) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}$$

Now

$$\mathbf{A}^{-1}(\mathbf{AC}) = \left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{C} = \mathbf{IC} = \mathbf{C}$$

and

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 4 & 4 \\ 12 & 11 \end{bmatrix}$$

So

$$\mathbf{C} = \begin{bmatrix} 5 & 5 \\ 4 & 4 \\ 12 & 11 \end{bmatrix}$$

$\square$