Math 3013
Solutions to Problem Set 2

1. Reduce the following matrices to row-echelon form, and reduced row-echelon form.

(a)

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 2 \\ 3 & -1 & 6 \end{bmatrix}$$

- Multiplying the first row by, respectively, $-\frac{1}{2}$ and $-\frac{3}{2}$ and, respectively, adding the results to the second and third rows produces

$$\begin{array}{l} R_2 \to R_2 - \frac{1}{2}R_1 \\ R_3 \to R_3 - \frac{3}{2}R_1 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 & 4 \\ 0 & \frac{5}{2} & 0 \\ 0 & -\frac{5}{2} & 0 \end{bmatrix}$$

Adding the second row to the third now yields

$$R_3 \to R_3 + R_2 \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 & 4 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in **row-echelon form**.

Multiplying the first row by $\frac{1}{2}$ and the second row by $\frac{2}{5}$ produces

$$\begin{array}{l} R_1 \to \frac{1}{2}R_1 \\ R_2 \to \frac{2}{3}R_2 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Adding $-\frac{1}{2}$ times the second row to the first row produces

$$R_1 \to R_1 - \frac{1}{2}R_2 \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in **reduced row-echelon form**. □

(b)

$$\begin{bmatrix} 0 & 2 & -1 & 3 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & -3 & 3 \\ 1 & 5 & 5 & 9 \end{bmatrix}$$

- Interchanging the first and second rows yields

$$R_1 \longleftrightarrow R_2 \quad \Rightarrow \quad \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & -1 & 3 \\ 1 & 1 & -3 & 3 \\ 1 & 5 & 5 & 9 \end{bmatrix}$$

Adding the first row to third and fourth rows yields

$$\begin{array}{l} R_3 \to R_3 + R_1 \\ R_4 \to R_4 + R_1 \end{array} \quad \Rightarrow \quad \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & -1 & 3 \\ 0 & 2 & -1 & 3 \\ 0 & 6 & 7 & 9 \end{bmatrix}$$

Adding, respectively, -1 and -3 times the second row to, respectively, the third and fourth row yields

$$\begin{array}{c} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 3R_2 \end{array} \quad \Rightarrow \quad \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix}$$

Interchanging the last two rows yields

$$R_3 \longleftrightarrow R_4 \quad \Rightarrow \quad \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in **row-echelon form**.

Multiplying row 1 by -1, row 2 by $\frac{1}{2}$, and row 3 by $\frac{1}{10}$, yields

$$\begin{array}{c} R_1 \to -R_1 \\ R_2 \to \frac{1}{2}R_2 \\ R_3 \to \frac{1}{10}R_3 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Adding 2 times the third row to the first row and $\frac{1}{2}$ times the third row to the second yields

$$\begin{array}{c} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 + \frac{1}{2}R_3 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Replacing the first row with its sum with the second row yields

$$R_1 \to R_1 + R_2 \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in **reduced row-echelon form**. □

(c)

$$\begin{bmatrix} -1 & 3 & 0 & 1 & 4 \\ 1 & -3 & 0 & 0 & -1 \\ 2 & -6 & 2 & 4 & 0 \\ 0 & 0 & 1 & 3 & -4 \end{bmatrix}$$

- Replacing the second and third rows, respectively, by their sums with the first row and 2 times the first row yields

$$\begin{array}{c} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 2R_1 \end{array} \quad \Rightarrow \quad \begin{bmatrix} -1 & 3 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 0 & 1 & 3 & -4 \end{bmatrix}$$

Replacing the last row by its sum with $-\frac{1}{2}$ times the third row yields

$$R_4 \to R_4 - \frac{1}{2}R_3 \quad \Rightarrow \quad \begin{bmatrix} -1 & 3 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 0 & 0 & 0 & -8 \end{bmatrix}$$

Interchanging the second and third rows yields

$$R_2 \longleftrightarrow R_3 \quad \Rightarrow \quad \begin{bmatrix} -1 & 3 & 0 & 1 & 4 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & -8 \end{bmatrix}$$

This last matrix is in **row-echelon form**.

Multiplying the first row by -1, the second row by $\frac{1}{2}$, and the fourth row by $-\frac{1}{8}$ yields

$$\begin{matrix} R_1 \to -R_1 \\ R_2 \to \frac{1}{2}R_2 \\ R_4 \to -\frac{1}{8}R_4 \end{matrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & -3 & 0 & -1 & -4 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Adding, 4 times, -4 times, and -3 times the last row to, respectively, rows 1, 2, and 3 yields

$$\begin{matrix} R_1 \to R_1 + 4R_4 \\ R_2 \to R_2 - 4R_4 \\ R_3 \to R_3 - 3R_4 \end{matrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Adding, 1 times and -3 times the third row to, respectively, the first and second rows yields

$$\begin{matrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 3R_3 \end{matrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This last matrix is in **reduced row-echelon form**. □

2. For each linear system below, construct the corresponding augmented matrix

(a)

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ x_2 + x_3 + 3x_4 &= -2 \end{aligned}$$

- $$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \end{array}\right]$$

(b)

$$\begin{aligned} x_5 &= 3 \\ x_1 + x_3 + x_4 &= 2 \\ x_2 - x_5 &= 1 \end{aligned}$$

- $$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 1 & 3 \\ 1 & 0 & 3 & 1 & 2 & \\ 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right]$$

3. Describe all the solutions of a linear system whose corresponding augmented matrix can be row reduced to the given matrix.

(a)

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Note that there are two pivots and four columns in the first block. According to the theorem at the end of Lecture 5 (Theorem 1.7 in the text) this meas that the number of free variables will be $4 - 2$. To see this more explicitly, let's write down the corresponding linear system

(1)
$$\begin{aligned} x_1 + 2x_3 &= 1 \\ x_2 + x_3 + 3x_4 &= -2 \\ 0 &= 0 \end{aligned}$$

The last equation is of no consequence. However, the first two equations can be used, respectively, to express x_1 and x_2 in terms of x_3 and x_4 :

(2)
$$\begin{aligned} x_1 &= 1 - 2x_3 \\ x_2 &= -2 - x_3 - 3x_4 \end{aligned}$$

But there is nothing left to determine x_3 and x_4. Thus we have two free variables: so long as we use equations (2) to determine x_1 and x_2, we can use any values we want for x_3 and x_4 and we'll still satisfy (1).

Let's now write down the form of a typical solution vector

$$\mathbf{x} = \begin{bmatrix} 1 - 2x_3 \\ -2 - x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

This exhibits our solutions as forming a 2-dimensional hyperplane in $\mathbb{R}^4$.

□

(b)

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- This augmented matrix is equivalent to the following system of linear equations

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_5 &= 1 \\ x_4 + 4x_5 &= 2 \\ 0 &= -1 \\ 0 &= 0 \end{aligned}$$

The fourth equation is obviously a contradiction. Hence, there is no solution to the linear system corresponding to this augmented matrix. This situation is of course predicted by Part 1 of the theorem at the end of Lecture 5 (Theorem 1.7 in the text). □

4. Find all solutions of the given linear system.

(a)

$$\begin{aligned} 2x - y &= 8 \\ 6x - 5y &= 32 \end{aligned}$$

- The corresponding augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -1 & 8 \\ 6 & -5 & 32 \end{array}\right]$$

Replacing the second row with its sum with -3 times the first yields

$$R_2 \to R_2 - 3R_1 \quad \Rightarrow \quad \left[\begin{array}{cc|c} 2 & -1 & 8 \\ 0 & -2 & 8 \end{array}\right]$$

Above is the augemented matrix in row echelon form. Let's continue to row reduce to reduced-row-echelon-form.

$$\begin{array}{c} R_1 \to \frac{1}{2}R_1 \\ R_2 \to -\frac{1}{2}R_2 \end{array} \quad \Rightarrow \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 4 \\ 0 & 1 & -4 \end{array}\right]$$

$$R_1 \to R_1 + \frac{1}{2}R_2 \quad \Rightarrow \quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -4 \end{array}\right]$$

This augmented matrix is in reduced-row-echelon-form; it will be the augmented matrix of our solution equations. Converting back to equations we get

$$\begin{aligned} x &= 2 \\ y &= -4 \end{aligned}$$

□

(b)

$$\begin{aligned} y + z &= 6 \\ 3x - y + z &= -7 \\ x + y - 3z &= -13 \end{aligned}$$

- The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 6 \\ 3 & -1 & 1 & -7 \\ 1 & 1 & -3 & -13 \end{array}\right]$$

Let's row reduce this augmented matrix to reduced-row-echelon-form.

$$R_1 \longleftrightarrow R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 3 & -1 & 1 & -7 \\ 0 & 1 & 1 & 6 \\ 1 & 1 & -3 & -13 \end{array}\right]$$

$$R_3 \to R_3 - \frac{1}{3}R_1 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 3 & -1 & 1 & -7 \\ 0 & 1 & 1 & 6 \\ 0 & \frac{4}{3} & -\frac{10}{3} & -\frac{32}{3} \end{array}\right]$$

$$R_3 \to R_3 - \frac{4}{3}R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 3 & -1 & 1 & -7 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -\frac{14}{3} & -\frac{56}{3} \end{array}\right]$$

This augmented matrix is in row-echelon form.

$$\begin{array}{c} R_1 \to \frac{1}{3}R_1 \\ R_3 \to -\frac{3}{14}R_3 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{7}{3} \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{c} R_1 \to R_1 - \frac{1}{3}R_3 \\ R_2 \to R_2 - R_3 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & 0 & -\frac{11}{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$R_1 \to R_1 + \frac{1}{3}R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

We've now arrived at the augmented matrix in reduced-row-echelon-form. The corresponding equations (of the solution) are

$$\begin{aligned} x &= -3 \\ y &= 2 \\ z &= 4 \end{aligned}$$

□

5. Determine whether the vector

$$\mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}$$

is in the span of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \quad , \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \quad , \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix}$$

- By definition, if $\mathbf{b}$ lies in the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, then

(3) $$\mathbf{b} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3$$

for some choice of coefficients x_1, x_2 and x_3. In terms of components, this vector equation is

$$\begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} x_2 - 3x_3 \\ 2x_1 + 4x_2 - x_3 \\ 4x_1 - 2x_2 + 5x_3 \end{bmatrix}$$

or

$$\begin{aligned} x_2 - 3x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 5 \\ 4x_1 - 2x_2 + 5x_3 &= 3 \end{aligned}$$

The augmented matrix for this linear system is

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & 3 \\ 2 & 4 & -1 & 5 \\ 4 & -2 & 5 & 3 \end{array}\right]$$

Interchanging the first and second rows yields

$$R_1 \longleftrightarrow R_1 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 2 & 4 & -1 & 5 \\ 0 & 1 & -3 & 3 \\ 4 & -2 & 5 & 3 \end{array}\right]$$

Replacing the third row with its sum with -2 times the first row yields

$$R_3 \to R_3 - 2R_1 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 2 & 4 & -1 & 5 \\ 0 & 1 & -3 & 3 \\ 0 & -10 & 7 & -7 \end{array}\right]$$

Replacing the third row with its sum with 10 times the second row yields

$$R_3 \to R_3 + 10R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|c} 2 & 4 & -1 & 5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 23 & -23 \end{array}\right]$$

According the theorem at the end of Lecture 5, the linear system corresponding to such an augmented matrix does indeed have a solution (in fact, a unique solution).

We conclude that we can find values of x_1, x_2, and x_3 such that (3) holds. Hence, $\mathbf{b}$ is in the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. □