Math 3013
Solutions to Problem Set 1

1. Let $\mathbf{u} = [1,2,1,0]$, $\mathbf{v} = [-2,0,1,6]$ and $\mathbf{w} = [3,-5,1,-2]$. Compute $\mathbf{u} - 2\mathbf{v} + 4\mathbf{w}$.

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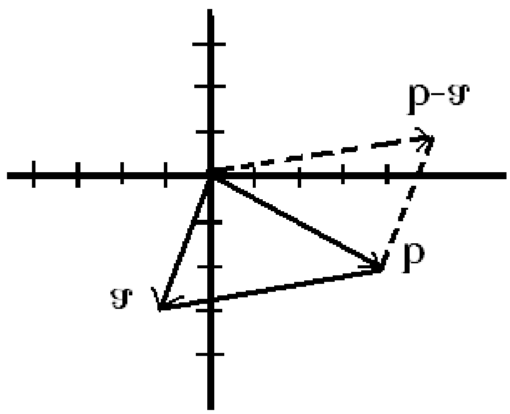
$$\begin{aligned}
\mathbf{u} - 2\mathbf{v} + \mathbf{w} &= [1,2,1,0] - 2[-2,0,1,6] + 4[3,-5,1,-2] \\
&= [1,2,1,0] + [4,0,-2,-12] + [12,-20,4,-8] \\
&= [1+4+12, 2+0-20, 1-2+4, 0-12-8] \\
&= [17,-18,3,-20]
\end{aligned}$$

□

2. Find the vector which, when translated, represents geometrically an arrow reaching from the point $(-1,3)$ to the point $(4,2)$ in $\mathbb{R}^2$.

- Set $\mathbf{a} = (-\mathbf{1},\mathbf{3})$ and $\mathbf{b} = (4,2)$. Then we have the following picture



The desired vector is a parallel transport of the vector $\mathbf{b} - \mathbf{a} = [4,2] - [-1,3] = [5,-1]$. □

3. Let $\mathbf{u} = [-1,3,4]$ and $\mathbf{v} = [2,1,-1]$. Compute $\|-\mathbf{u}\|$ and $\|\mathbf{v}+\mathbf{u}\|$.

- We have

$$\|-\mathbf{u}\| = \sqrt{(-\mathbf{u})\cdot(-\mathbf{u})} = \sqrt{(-1)^2 + (3)^2 + (4)^3} = \sqrt{1+9+16} = \sqrt{26}$$

$$\mathbf{u}+\mathbf{v} = [-1+2, 3+1, 4-1] = [1,4,3] \quad \Rightarrow \quad \|\mathbf{u}+\mathbf{v}\| = \sqrt{1^2+4^2+3^2} = \sqrt{26}$$

4. Compute the angle between $[1,-1,2,3,0,4]$ and $[7,0,1,3,2,4]$ in $\mathbb{R}^6$.

- Let $\mathbf{a} = [1,-1,2,3,0,4]$ and $\mathbf{b} = [7,0,1,3,2,4]$. From the geometric interpretation of the dot product, we have

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos(\theta)$$

or

$$\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$$

or

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right)$$

Now

$$\begin{aligned}\|\mathbf{a}\| &= \sqrt{(1)^2+(-1)^2+(2)^2+(3)^2+(0)^2+(4)^2} = \sqrt{31} \\ \|\mathbf{b}\| &= \sqrt{(7)^2+(0)^2+(1)^2+(3)^2+(2)^2+(4)^2} = \sqrt{79}\end{aligned}$$

and

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (1)(7)+(-1)(0)+(2)(1)+(3)(3)+(0)(2)+(4)(4) \\ &= 7+0+2+9+16 \\ &= 34\end{aligned}$$

:So

$$\theta = \cos^{-1}\left(\frac{34}{\sqrt{31}\sqrt{79}}\right) = .81338 \text{ radians} = 46.603 \text{ degrees}$$

□

5. Prove that $(2,0,4)$, $(4,1,-1)$ and $(6,7,7)$ are the vertices of a right triangle in $\mathbb{R}^3$.

- We just need to show that two of the sides of triangle with vertices $\mathbf{A} = (2,0,4)$, $\mathbf{B} = (4,1,-1)$ and $\mathbf{C} = (6,7,7)$ are perpendicular. However, the vectors $\mathbf{A}$,$\mathbf{B}$, and $\mathbf{C}$ are the vertices, not the sides. To get the sides we have to compute $\overline{\mathbf{AB}} = \mathbf{A}-\mathbf{B}$, $\overline{\mathbf{AC}} = \mathbf{A}-\mathbf{C}$, and $\overline{\mathbf{BC}} = \mathbf{B}-\mathbf{C}$:

$$\begin{aligned}\overline{\mathbf{AB}} &= \mathbf{A}-\mathbf{B} = (2,0,4)-(4,1,-1) = (-2,-1,5) \\ \overline{\mathbf{AC}} &= \mathbf{A}-\mathbf{C} = (2,0,4)-(6,7,7) = (-4,-7,-3) \\ \overline{\mathbf{BC}} &= \mathbf{B}-\mathbf{C} = (4,1,-1)-(6,7,7) = (-2,-6,-8)\end{aligned}$$

We then have

$$\begin{aligned}\overline{\mathbf{AB}}\cdot\overline{\mathbf{AC}} &= (-2)(-4)+(-1)(-7)+(5)(-3) = 8+7-15 = 0 \\ \overline{\mathbf{AB}}\cdot\overline{\mathbf{BC}} &= (-2)(-2)+(-1)(-6)+(5)(-8) = 4+6+40 = 50 \\ \overline{\mathbf{AC}}\cdot\overline{\mathbf{BC}} &= (-4)(-2)+(-4)(-6)+(-3)(-3) = 8+24+9 = 41\end{aligned}$$

Since $\overline{\mathbf{AB}}\cdot\overline{\mathbf{AC}} = 0$ we can conclude that these two sides are perpendicular, and the triangle is indeed a right triangle. □

6. Specify the line that passes through the points $(1,0,2)$ and $(2,1,0)$ as a set of vectors. (i.e. find vectors $\mathbf{p}_0$ and $\mathbf{v}$ such that line corresponds to the set $\ell = \{\mathbf{p}_0 + t\mathbf{v} \mid t \in \mathbb{R}\}$.

- We can guarantee that the point $(1,0,2)$ lies on the line

$$\ell = \{\mathbf{p}_0 + t\mathbf{v} \mid t \in \mathbb{R}\}$$

by setting

$$\mathbf{p}_0 = (1,0,2)$$

(and taking $t = 0$). We also want $(2,1,0)$ to live on the line. This we can do by forcing

$$(2,1,0) = (1,0,2) + t\mathbf{v} \quad \Rightarrow \quad (2,1,0)-(1.0,2) = t\mathbf{v}$$

The simplest way to solve the last equation is to let $t = 1$ and

$$\mathbf{v} = (2,1,0)-(1,0,2) = (1,1,2)$$

Thus, our line is

$$\ell = \{(1,0,2) + t\,(1,1,2) \mid t \in \mathbb{R}\}$$

7. Let

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 2 & -1 \\ 0 & 6 \\ -3 & 2 \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix}$$

(a) $3\mathbf{A}$

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$$3\mathbf{A} = 3\begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 3 & 9 \\ 12 & 0 & -3 \end{bmatrix}$$

□

(b) $\mathbf{A} + \mathbf{B}$

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$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -2+4 & 1+1 & 3+-2 \\ 4+5 & 0+-1 & -1+3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 9 & -1 & 2 \end{bmatrix}$$

□

(c) $\mathbf{AB}$

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$$\mathbf{AB} = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix}$$

Which is undefined since the number of columns in the first matrix is not the same as the number of rows in the second matrix. □

(d) $\mathbf{A}^2$

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$$\mathbf{A}^2 = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}$$

Which is undefined since the number of columns in the first matrix is not the same as the number of rows in the second matrix. □

(e) $(2\mathbf{A} - \mathbf{B})\,\mathbf{D}$

- $$
\begin{aligned}
(2\mathbf{A}-\mathbf{B})\,\mathbf{D} &= \left(2\begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix}\right)\begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix} \\
&= \left(\begin{bmatrix} -4 & 2 & 6 \\ 8 & 0 & -2 \end{bmatrix} + \begin{bmatrix} -4 & -1 & 2 \\ -5 & 1 & -3 \end{bmatrix}\right)\begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix} \\
&= \left(\begin{bmatrix} -8 & 1 & 8 \\ 3 & 1 & -5 \end{bmatrix}\right)\begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix} \\
&= \begin{bmatrix} (-8,1,8)\cdot(-4,3,-1) & (-8,1,8)\cdot(2,5,-3) \\ (3,1,-5)\cdot(-4,3,-1) & (3,1,-5)\cdot(2,5,-3) \end{bmatrix} \\
&= \begin{bmatrix} (-8)(-4)+(1)(3)+(8)(-1) & (-8)(2)+(1)(5)+(8)(-3) \\ (3)(-4)+(1)(3)+(-5)(-1) & (3)(2)+(1)(5)+(-5)(-3) \end{bmatrix} \\
&= \begin{bmatrix} 27 & -35 \\ -4 & 26 \end{bmatrix}
\end{aligned}
$$

□

(f) **ADB**

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\begin{aligned}
\mathbf{ADB} &= \mathbf{A}\,(\mathbf{DB}) = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}\left(\begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix}\right) \\
&= \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} (-4,2)\cdot(4,5) & (-4,2)\cdot(1,-1) & (-4,2)\cdot(-2,3) \\ (3,5)\cdot(4,5) & (3,5)\cdot(1,-1) & (3,5)\cdot(-2,3) \\ (-1,-3)\cdot(4,5) & (-1,-3)\cdot(1,-1) & (-1,-3)\cdot(-2,3) \end{bmatrix} \\
&= \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} -16+10 & -4-2 & 8+6 \\ 12+25 & 3-5 & -6+15 \\ -4-15 & -1+3 & 2-9 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} -6 & -6 & 14 \\ 37 & -2 & 9 \\ -19 & 2 & -7 \end{bmatrix} \\
&= \begin{bmatrix} (-2,1,3)\cdot(-6,37,-19) & (-2,1,3)\cdot(-6,-2,2) & (-2,1,3)\cdot(14,9,-7) \\ (4,0,-1)\cdot(-6,37,-19) & (4,0,-1)\cdot(-6,-2,2) & (4,0,-1)\cdot(14,9,-7) \end{bmatrix} \\
&= \begin{bmatrix} 12+37-57 & 12-2+6 & -28+9-21 \\ -24+0+19 & -24+0-2 & 56+0+7 \end{bmatrix} \\
&= \begin{bmatrix} -8 & 16 & -40 \\ -5 & -26 & 63 \end{bmatrix}
\end{aligned}
$$

□

8. Consider the row and column vectors

$$
\mathbf{x} = [-2,3,-1] \quad , \quad \mathbf{y} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}
$$

Compute the matrix products **xy** and **yx**.

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$$\begin{aligned}
\mathbf{xy} &= [-2,3,-1]\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = (-2,3,-1)\cdot(4,-1,3) = -14 \\
\mathbf{yx} &= \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}[-2,3,-1] = \begin{bmatrix} -8 & 12 & -4 \\ 2 & -3 & 1 \\ -6 & 9 & -3 \end{bmatrix}
\end{aligned}$$

$\square$