Math 3013

SOLUTIONS TO SAMPLE SECOND EXAM

1. Write down the formal definitions of the following notions:

(a) a *linear transformation from* $\mathbb{R}^m$ to $\mathbb{R}^n$

- A linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ is a function $T:\mathbb{R}^m\to\mathbb{R}^n$ such that

$$\begin{aligned} T(\mathbf{x}_1+\mathbf{x}_2) &= T(\mathbf{x}_1)+T(\mathbf{x}_2) \qquad \text{for all } \mathbf{x}_1,\mathbf{x}_2\in\mathbb{R}^m \\ T(\lambda\mathbf{x}) &= \lambda T(\mathbf{x}) \qquad \text{for all } \lambda\in\mathbb{R} \text{ and all } \mathbf{x}\in\mathbb{R}^m \end{aligned}$$

(b) the *range* of a linear transfomation $T:\mathbb{R}^m\to\mathbb{R}^n$

- The range of a linear transformation $T:\mathbb{R}^m\to\mathbb{R}^n$ is the set

$$Range(T)=\{\mathbf{y}\in\mathbb{R}^n \mid \mathbf{y}=T(\mathbf{x}) \text{ for some } \mathbf{x}\in\mathbb{R}^m\}\subset\mathbb{R}^n$$

(c) the *kernel* of a linear transformation $T:\mathbb{R}^m\to\mathbb{R}^n$

- The kernel of a linear transformation $T:\mathbb{R}^m\to\mathbb{R}^n$ is the set

$$Ker(T)=\{\mathbf{x}\in\mathbb{R}^m \mid T(\mathbf{x})=\mathbf{0}\}\subset\mathbb{R}^m$$

2. Consider the following mapping: $T:\mathbb{R}^3\to\mathbb{R}^2 : T([x_1,x_2,x_3])=[x_2,x_1-x_3]$. Show that T is a linear transformation.

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$$T(\lambda[x_1,x_2,x_3])=T([\lambda x_1,\lambda x_2,\lambda x_3])=[\lambda x_2,\lambda x_1-\lambda x_3]=\lambda[x_2,x_1-x_3]=\lambda T([x_1,x_2,x_3]) \quad\Rightarrow\quad T(\lambda\mathbf{x})=\lambda T(\mathbf{x})$$

$$T(\mathbf{x}+\mathbf{x}')=T([x_1+x_1',x_2+x_2',x_3+x_3'])=[x_2+x_2',(x_1+x_1')-(x_3+x_3')]=[x_2,x_1-x_3]+[x_2',x_1'-x_3']=T(\mathbf{x})+T(\mathbf{x}')$$

Since T preserves scalar multiplication and vector addition, T is a linear transformation. $\square$

3. Suppose T is the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ given by

$$T([x_1,x_2,x_3])=[x_1+x_2\ ,\ -x_1+x_3\ ,\ x_2+x_3\ ,\ 0]$$

(a) Find the matrix $\mathbf{A}_T$ such that $T(\mathbf{x})=\mathbf{A}\mathbf{x}$ for all $\mathbf{x}\in\mathbb{R}^3$.

- We first calculate the action of T on the standard basis vectors for the domain $\mathbb{R}^3$:

$$\begin{aligned} T(\mathbf{e}_1) &= T([1,0,0])=[1,-1,0,0] \\ T(\mathbf{e}_2) &= T([0,1,0])=[1,0,1,0] \\ T(\mathbf{e}_3) &= T([0,0,1])=[0,1,1,0] \end{aligned}$$

Converting these to columns gives us the matrix $\mathbf{A}_T$

$$\mathbf{A}_T=\begin{pmatrix} \uparrow & \uparrow & \uparrow \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}=\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) Find a basis for the *range* of T

- The range of T is equivalent to the column space of $\mathbf{A}_T$. To find the latter we first row reduce $\mathbf{A}_T$ to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since we only have pivots in the first two columns of the row echelon form, the first two columns of $\mathbf{A}_T$ will provide a basis for the column space of $\mathbf{A}_T$, and so also (once reinterpreted as vectors in $\mathbb{R}^4$) a basis for the range of T :

$$\text{basis for } ColSp\left(\mathbf{A}_T\right) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \Rightarrow \quad \text{basis for } Range\left(T\right) = \left\{ [1,-1,0,0] \ , \ [1,0,1,0] \right\}$$

(c) Find a basis for the *kernel* of T.

- The kernel of T will correspond to the null space of the matrix $\mathbf{A}_T$ (i.e., the solution set of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$). Since we have already row reduced $\mathbf{A}_T$ to a reduced row echelon form in part (b) above, we can use that RREF for $\mathbf{A}_T$ to determine a basis for the null space of $\mathbf{A}_T$:

$$NullSp\left(\mathbf{A}_T\right) = NullSp \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{solution set of} \left. \begin{array}{r} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{array} \right\}$$

Since the third column of the RREF does not contain a pivot, x_3 is to be regarded as a free parameter. Writing the general solution vector in terms of the free parameter we get

$$\mathbf{x} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

We can now conclude

$$\text{basis for } NullSp\left(\mathbf{A}_t\right) = \left\{ \begin{matrix} 1 \\ -1 \\ 1 \end{matrix} \right\} \quad \Rightarrow \quad \text{basis for } Ker\left(T\right) = \left\{ [1,-1,1] \right\}$$

4. Compute the following determinants by the indicated method

(a) $\det \begin{pmatrix} 3 & 3 & 0 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ via row reduction

- We have

$$\begin{aligned} \det \begin{pmatrix} 3 & 3 & 0 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \xrightarrow{R_3 \longleftrightarrow R_1} -\det \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 0 & 1 \\ 3 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ & \xrightarrow{R_3 \to R_3 - 3R_1} -\det \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -6 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ & = -\left(1\right)\left(2\right)\left(-6\right)\left(1\right) \\ & = 12 \end{aligned}$$

(the sign flip because we interchanged rows)

(b) $\det \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ via a cofactor expansion

- Cofactor expansion along the second row:

$$\begin{aligned}\det\begin{pmatrix}1 & -1 & 1\\ 0 & 0 & 2\\ 1 & 1 & 1\end{pmatrix} &= (0)(-1)^{2+1}\det\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} + (0)(-1)^{2+1}\det\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + (2)(-1)^{2+3}\det\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\ &= 0+0+(-2)(2)\\ &= 4\end{aligned}$$

5. For each of the matrices $\mathbf{A}$ below

- Find the characteristic polynomial $p_{\mathbf{A}}(\lambda)$ of $\mathbf{A}$
- Find the eigenvalues of $\mathbf{A}$
- Find the eigenvectors of $\mathbf{A}$
- Determine the both algebraic muliplicites and geometric multiplicity of each eigenvalue

(a) $\mathbf{A} = \begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$

- We have

$$\begin{aligned}p_{\mathbf{A}}(\lambda) &= \det(\mathbf{A}-\lambda\mathbf{I}) = \det\begin{pmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{pmatrix} = (1-\lambda)^2 - 4\\ &= \lambda^2 - 2\lambda + 1 - 4 = \lambda^2 - 2\lambda - 3\\ &= (\lambda-3)(\lambda+1)\end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of $p_{\mathbf{A}}(\lambda) = 0$. These are obviously

$$\lambda = 3, 1$$

We'll now find the corresponding eigenvectors:

$\lambda = 3$: we need to find a basis for $NullSp(\mathbf{A}-3\mathbf{I}) = NullSp\begin{pmatrix}-2 & 2\\ 2 & -2\end{pmatrix} = NullSp\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix} = span\left(\begin{bmatrix}1\\ 1\end{bmatrix}\right)$. Thus,

$$\text{3-eigenspace } = span\left(\begin{bmatrix}1\\ 1\end{bmatrix}\right)$$

$\lambda = -1$: we need to find a basis for $NullSp(\mathbf{A}+\mathbf{I})) = NullSp\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix} = NullSp\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} = span\left(\begin{bmatrix}-1\\ 1\end{bmatrix}\right)$. Thus,

$$-1\text{-eigenspace } = span\left(\begin{bmatrix}-1\\ 1\end{bmatrix}\right)$$

Since $p_{\mathbf{A}}(\lambda)$ has exactly one factor of $(\lambda-3)$ and exactly one factor of $(\lambda+1)$, the algebraic multiplicities of both eigenvalues is 1.

Since both the $\lambda = 3$ and $\lambda = -1$ eigenspaces have exactly one basis vector, the geometric multiplicities of both eigenvalues is also 1.

(b) $\mathbf{A} = \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$

- The characteristic polynomial is

$$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A}-\lambda\mathbf{I}) = (\lambda-1)^3$$

We thus have only one eigenvalue $\lambda = 1$. Note that it has algebraic multiplicity 3.

To find a basis for corresponding eigenspace we find a basis for

$$NullSp\left(\mathbf{A}_T - \mathbf{I}\right) = NullSp\left(\left(\begin{array}{ccc} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)\right) = NullSp\left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

Evidently, the null space will consist of solution vectors of the form $\left[\begin{array}{c} 0 \\ 0 \\ x_3 \end{array}\right] \in span\left(\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]\right)$.

We thus have

$$\text{1-eigenspace } = span\left(\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]\right)$$

The geometric multiplicity of $\lambda = 1$ is 1 since we have only one basis vector for the 1-eigenspace.