LECTURE 15

Coordinatization and Change of Basis

1. Review: Coordinatization of Vectors

Suppose \( B = \{u_1, u_2, \ldots, u_n\} \) is a basis of \( \mathbb{R}^n \). Then, by definition, for any vector \( v \in \mathbb{R}^n \), there is a unique choice of coefficients \( c_1, \ldots, c_n \) such that

\[
v = c_1 u_1 + \cdots + c_n u_n
\]

We can thus take the (ordered set of) \( n \) numbers \( c_1, \ldots, c_n \) and assemble them into a vector. We call the result the coordinate vector of \( u \) with respect to the basis \( B \), and denote it by \( v_B \):

\[
v_B = [c_1, \ldots, c_n]
\]

Example 15.1. Find the coordinate vector of \([1, 2, -2]\) with respect to the basis \( \{[1, 1, 1], [1, 2, 0], [1, 0, 1]\} \) of \( \mathbb{R}^3 \).

• We need to solve

\[
\begin{align*}
1 & 1 & 1 \\
1 & 2 & 0 \\
1 & 0 & 1 \\
\end{align*}
\begin{align*}
c_1 & + c_2 & + c_3 = 1 \\
1 & 2 & 0 & 2 \\
1 & 0 & 1 & -2 \\
\end{align*}
\]

which is the linear system corresponding to the following augmented matrix:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 0 & 2 \\
1 & 0 & 1 & -2 \\
\end{bmatrix}
\]

This matrix row reduces to the following reduced row echelon form:

\[
\begin{bmatrix}
1 & 0 & 0 & -4 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 2 \\
\end{bmatrix}
\]

And so the solution is

\[
c_1 = -4, \quad c_2 = 3, \quad c_3 = 2
\]

So

\[
v_B = [-4, 3, 2]
\]

Summary: To find the coordinate vector \( v_B \) of a vector \( v \in \mathbb{R}^n \) with respect to an ordered basis \( B = \{u_1, \ldots, u_n\} \),

Fact 15.2. (1) Form the augmented matrix \([u_1, \ldots, u_n \mid v]\) using the vectors \( u_1, \ldots, u_n \) as the first \( n \) columns, and the vector \( v \) as the last column.

(2) Row reduce the augmented matrix to reduced row-echelon form \([I \mid v_B]\). The vector \( v_B \) in the last column will be the desired coordinate vector.

The method described above is good for finding the coordinate vector of a particular vector \( v \) with respect to the basis \( B = \{u_1, \ldots, u_n\} \), but what if we had needed to find the coordinate vectors for a whole bunch
of vectors \( \mathbf{v} \) or even an arbitrary vector \( \mathbf{v} \) in \( \mathbb{R}^n \). Well, note that the left hand side of the condition we wish to satisfy

\[
c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \ldots + c_n \mathbf{u}_n = \mathbf{v}
\]

can be written

\[
\mathbf{M}_B \mathbf{c} = \mathbf{v}
\]

where \( \mathbf{M} \) is the matrix constructed by using the basis vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) as columns and \( \mathbf{c} \) is the column vector constructed out of the coefficients \( c_1, \ldots, c_n \) when we apply the identity

\[
\mathbf{M}_B \mathbf{c} = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} + c_2 \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} + \cdots + c_n \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix}
\]

Thus,

**Fact 15.3.** The problem of finding the coordinate vector \( \mathbf{v}_B \) for a vector \( \mathbf{v} \) with respect to a basis \( B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \) is equivalent to solving the following linear system

\[
\mathbf{M}_B \mathbf{x} = \mathbf{v}
\]

Now note that since the elements of a basis are, practically by definition, linearly independent, the matrix \( \mathbf{M}_B \) constructed as above from a basis \( B \) will always be invertible. Thus, for any vector \( \mathbf{v} \) we can obtain the coordinate vector \( \mathbf{v}_B \) of \( \mathbf{v} \) with respect to \( B \) by setting

\[
\mathbf{v}_B = (\mathbf{M}_B)^{-1} \mathbf{v}
\]

**Reinterpretation:**

Think of the original vector \( \mathbf{v} = [v_1, \ldots, v_n] \) as the coordinate vector of \( \mathbf{v} \) with respect to the standard basis \( B_{std} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \) of \( \mathbb{R}^n \):

\[
\mathbf{v} = [v_1, \ldots, v_n] = v_1 [1,0,\ldots,0] + v_2 [0,1,0,\ldots,0] + \cdots + v_n [0,\ldots,0,1] \]

\[
\Rightarrow \quad \mathbf{v}_{B_{std}} = [v_1, \ldots, v_n]
\]

Then the matrix \( (\mathbf{M}_B)^{-1} \) converts the coordinate vector for \( \mathbf{v} \) with respect to the standard basis to the coordinate vector for \( \mathbf{v} \) with respect to the basis \( B \):

\[
\mathbf{v}_B = (\mathbf{M}_B)^{-1} \mathbf{v} = (\mathbf{M}_B)^{-1} \mathbf{v}_{B_{std}}
\]

### 2. Change of Basis

Suppose now we have two different, non-standard, bases for \( \mathbb{R}^n \):

\[
B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}
\]

\[
B' = \{\mathbf{b}_1', \ldots, \mathbf{b}_n'\}
\]

Let \( \mathbf{v} \) be an arbitrary vector in \( \mathbb{R}^n \). How might we convert the coordinate vector \( \mathbf{v}_B \) for \( \mathbf{v} \) with respect to the basis \( B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \) directly to the coordinate vector \( \mathbf{v}_{B'} \) for \( \mathbf{v} \) with respect to the basis \( B' \). Well, let

\[
\mathbf{M}_B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}, \quad \mathbf{M}_{B'} = \begin{bmatrix} \mathbf{b}_1' & \cdots & \mathbf{b}_n' \end{bmatrix}
\]

so that

\[
\mathbf{v}_B = (\mathbf{M}_B)^{-1} \mathbf{v}
\]

\[
\mathbf{v}_{B'} = (\mathbf{M}_{B'})^{-1} \mathbf{v}
\]
Multiplying the first equation from the left by $M_B$ we get $M_B v_B = v$. Substituting this into the second equation yields
\[
v_{B'} = (M_{B'})^{-1} M_B v_B
\]
If we now set
\[
C_{B,B'} = M_{B'}^{-1} M_B
\]
then multiplying any coordinate vector $v_B$ (with respect to the basis $B$) by $C_{B,B'}$ produces the corresponding coordinate vector $v_{B'}$ (with respect to the basis $B'$). We call $C_{B,B'}$ the change-of-coordinates matrix. It is the matrix that converts coordinate vectors expressed in terms of the ordered basis $B$ to the coordinate vectors with respect to the ordered basis $B'$.

2.1. Calculating $C_{B,B'}$.
\[
\left[ \begin{array}{ccc|ccc} b'_1 & \cdots & b'_n \\ \vdots & \ddots & \vdots \\ b_n & & b_n \end{array} \right] \left[ \begin{array}{ccc} b_1 & \cdots & b_n \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} b_1 & \cdots & b_n \end{array} \right]
\]
To see why this works, first recall that the sequence of row operations that reduce reduce the matrix $M_{B'}$ to the identity can be also be carried out by applying a product of elementary matrices; and that the product of these elementary matrices is exactly $M_{B'}^{-1}$. Visually, if
\[
I = R_k (R_{k-1} \cdots (R_1 (M_{B'})))
\]
then
\[
(M_{B'})^{-1} = E_{R_k} E_{R_{k-1}} \cdots E_{R_1}
\]
and so
\[
\begin{align*}
R_k \left( R_{k-1} \cdots (R_1 \left( \left[ \begin{array}{ccc|ccc} b'_1 & \cdots & b'_n \\ \vdots & \ddots & \vdots \\ b_n & & b_n \end{array} \right] \left[ \begin{array}{ccc} b_1 & \cdots & b_n \end{array} \right] \right) \right) & = \left[ E_{R_k} E_{R_{k-1}} \cdots E_{R_1} M_{B'} \right] \left[ E_{R_k} E_{R_{k-1}} \cdots E_{R_1} M_B \right] \\
& = \left[ (M_{B'})^{-1} M_B \right] \left( M_{B'} \right) \\
& \equiv \left[ I \mid C_{B,B'} \right]
\end{align*}
\]

Example 15.4. Calculate the change-of-coordinates matrix taking the basis $B = \{[1, -1], [1, 1]\}$ to the basis $B' = \{[0, 1], [2, 1]\}$.

\[
\begin{bmatrix}
0 & 2 & 1 & 1 \\
1 & 1 & -1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & 2 & 1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & 1 & 1/2 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & -3/2 & 1/2 \\
0 & 1 & 1/2 & 1/2
\end{bmatrix}
\]
and so
\[
C_{B,B'} = \begin{bmatrix}
-3/2 & 1/2 \\
1/2 & 1/2
\end{bmatrix}
\]

2.2. General Change of Basis. Now let $V$ be an arbitrary vector space (not necessarily $\mathbb{R}^n$) with basis $B = \{b_1, \ldots, b_n\}$, and suppose $B' = \{b'_1, \ldots, b'_n\}$ is another basis for $V$. How can we find the coordinate vector $v_{B'}$ of a vector $v \in V$ if we only know its coordinates with respect to the basis $B$?

Well, this is the same sort of problem. The coordinate vectors $v_B$ and $v_{B'}$ are the vectors
\[
v_B = [c_1, \ldots, c_n] \in \mathbb{R}^n \\
v_{B'} = [c'_1, \ldots, c'_n] \in \mathbb{R}^n
\]
such that

\[ (*) \quad c_1 \mathbf{b}_1 + \cdots + c_n \mathbf{b}_n = \mathbf{v} = c'_1 \mathbf{b}'_1 + \cdots + c'_n \mathbf{b}'_n \]

Suppose we know the coordinate vectors of \( \mathbf{b}'_1, \ldots, \mathbf{b}'_n \) with respect to the basis \( B \).

\[
\begin{align*}
\mathbf{b}'_1 &= b'_{11} \mathbf{b}_1 + \cdots + b'_{1n} \mathbf{b}_n \\
& \vdots \\
\mathbf{b}'_n &= b'_{n1} \mathbf{b}_1 + \cdots + b'_{nn} \mathbf{b}_n
\end{align*}
\]

Then, since the vectors \( \mathbf{b}_1, \ldots, \mathbf{b}_n \) are all linearly independent, the linear system (*)

\[
c_1 \mathbf{b}_1 + \cdots + c_n \mathbf{b}_n = c'_1 (b'_{11} \mathbf{b}_1 + \cdots + b'_{1n} \mathbf{b}_n) + \cdots + c'_n (b'_{n1} \mathbf{b}_1 + \cdots + b'_{nn} \mathbf{b}_n)
\]

leads to

\[
\begin{align*}
c_1 &= (c'_1 b'_{11} + \cdots + c'_n b'_{n1}) \\
c_2 &= (c'_1 b'_{21} + \cdots + c'_n b'_{n2}) \\
& \vdots \\
c_n &= (c'_1 b_{1n} + \cdots + c'_n b'_{nn})
\end{align*}
\]

or

\[
\mathbf{v}_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b'_{11} & b'_{1n} \\ \vdots & \vdots \\ b'_{n1} & b_{nn} \end{bmatrix} \begin{bmatrix} c'_1 \\ \vdots \\ c'_n \end{bmatrix} = C_{BB'} \mathbf{v}_{B'}
\]