LECTURE 15

Coordinatization and Change of Basis

1. Review: Coordinatization of Vectors

Suppose $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis of $\mathbb{R}^n$. Then, by definition, for any vector $\mathbf{v} \in \mathbb{R}^n$, there is a unique choice of coefficients $c_1, \ldots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$$

We can thus take the (ordered set of) n numbers $c_1, \ldots, c_n$ and assemble them into a vector. We call the result the **coordinate vector of u with respect to the basis** B, and denote it by $\mathbf{v}_B$:

$$\mathbf{v}_B = [c_1, \ldots, c_n]$$

EXAMPLE 15.1. Find the coordinate vector of $[1, 2, -2]$ with respect to the basis $\{[1, 1, 1], [1, 2, 0], [1, 0, 1]\}$ of $\mathbb{R}^3$.

- We need to solve
$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$
which is the linear system corresponding to the following augmented matrix:
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 0 & 1 & -2 \end{array}\right]$$
This matrix row reduces to the following reduced row echelon form:
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$
And so the solution is
$$c_1 = -4 \quad , \quad c_2 = 3 \quad , \quad c_3 = 2$$
So
$$\mathbf{v}_B = [-4, 3, 2]$$

Summary: To find the coordinate vector $\mathbf{v}_B$ of a vector $\mathbf{v} \in \mathbb{R}^n$ with respect to an ordered basis $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$,

FACT 15.2.
(1) *Form the augmented matrix* $[\mathbf{u}_1, \ldots, \mathbf{u}_n \mid \mathbf{v}]$ *using the vectors* $\mathbf{u}_1, \ldots, \mathbf{u}_n$ *as the first* n *columns, and the vector* $\mathbf{v}$ *as the last column.*
(2) *Row reduce the augmented matrix to reduced row-echelon form* $[\mathbf{I} \mid \mathbf{v}_B]$. *The vector* $\mathbf{v}_B$ *in the last column will be the desired coordinate vector.*

The method described above is good for finding the coordinate vector of a *particular* vector $\mathbf{v}$ with respect to the basis $\mathbf{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, but what if we had needed to find the coordinate vectors for a whole bunch of vectors $\mathbf{v}$ or even an *arbitrary* vector $\mathbf{v}$ in $\mathbb{R}^n$. Well, note that the left hand side of the condition we wish to satisfy

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{v}$$

can be written

$$\mathbf{M}_B\mathbf{c} = \mathbf{v}$$

where $\mathbf{M}$ is the matrix constructed by using the basis vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as columns and $\mathbf{c}$ is the column vector constructed out of the coefficients $c_1, \ldots, c_n$ when we apply the identity

$$\mathbf{M}_B\mathbf{c} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{u}_1 & \cdots & \mathbf{u}_n \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 \begin{bmatrix} \uparrow \\ \mathbf{u}_1 \\ \downarrow \end{bmatrix} + c_2 \begin{bmatrix} \uparrow \\ \mathbf{u}_2 \\ \downarrow \end{bmatrix} + \cdots + c_n \begin{bmatrix} \uparrow \\ \mathbf{u}_n \\ \downarrow \end{bmatrix}$$

Thus,

FACT 15.3. *The problem of finding the coordinate vector $\mathbf{v}_B$ for a vector $\mathbf{v}$ with respect to a basis $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is equivalent to solving the following linear system*

$$\mathbf{M}_B\mathbf{x} = \mathbf{v}$$

Now note that since the elements of a basis are, practically by definition, linearly independent, the matrix $\mathbf{M}_B$ constructed as above from a basis B will always be invertible. Thus, for *any* vector $\mathbf{v}$ we can obtain the coordinate vector $\mathbf{v}_B$ of $\mathbf{v}$ with respect to B by setting

$$\mathbf{v}_B = (\mathbf{M}_B)^{-1}\mathbf{v} .$$

Reinterpretation:

Think of the original vector $\mathbf{v} = [v_1, \ldots, v_n]$ as the coordinate vector of $\mathbf{v}$ with respect to the *standard basis* $B_{std} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$:

$$\begin{aligned} \mathbf{v} &= [v_1, \ldots, v_n] = v_1 [1, 0, \ldots, 0] + v_2 [0, 1, 0, \ldots, 0] + \cdots v_n [0, \ldots, 0, 1] \\ &= v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{v}_n \\ \Longrightarrow &\quad \mathbf{v}_{B_{std}} = [v_1, \ldots, v_n] \end{aligned}$$

Then the matrix $(\mathbf{M}_B)^{-1}$ converts the coordinate vector for $\mathbf{v}$ with respect to the standard basis to the coordinate vector for $\mathbf{v}$ with respecto to the basis B:

$$\mathbf{v}_B = (\mathbf{M}_B)^{-1}\mathbf{v} = (\mathbf{M}_B)^{-1}\mathbf{v}_{B_{std}}$$

2. Change of Basis

Suppose now we have two different, non-standard, bases for $\mathbb{R}^n$:

$$\begin{aligned} B &= \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \\ B' &= \{\mathbf{b}'_1, \ldots, \mathbf{b}'_n\} \end{aligned}$$

Let $\mathbf{v}$ be an arbitrary vector in $\mathbb{R}^n$. How might we convert the coordinate vector $\mathbf{v}_B$ for $\mathbf{v}$ with respect to the basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ directly to the coordinate vector $\mathbf{v}_{B'}$ for $\mathbf{v}$ with respect to the basis B'. Well, let

$$\mathbf{M}_B = \begin{bmatrix} | & \cdots & | \\ \mathbf{b}_1 & \cdots & \mathbf{b}_n \\ | & \cdots & | \end{bmatrix} \quad , \quad \mathbf{M}_{B'} = \begin{bmatrix} | & \cdots & | \\ \mathbf{b}'_1 & \cdots & \mathbf{b}'_n \\ | & \cdots & | \end{bmatrix}$$

so that

$$\begin{aligned} \mathbf{v}_B &= (\mathbf{M}_B)^{-1}\mathbf{v} \\ \mathbf{v}_{B'} &= (\mathbf{M}_{B'})^{-1}\mathbf{v} \end{aligned}$$

Multiplying the first equation from the left by $\mathbf{M}_B$ we get $\mathbf{M}_B\mathbf{v}_B = \mathbf{v}$. Substituting this into the second equation yields

$$\mathbf{v}_{B'} = (\mathbf{M}_{B'})^{-1}\mathbf{M}_B\mathbf{v}_B$$

If we now set

$$\mathbf{C}_{B,B'} = \mathbf{M}_{B'}^{-1}\mathbf{M}_B$$

then multiplying any coordinate vector $\mathbf{v}_B$ (with respect to the basis B) by $\mathbf{C}_{B,B'}$ produces the corresponding coordinate vector $\mathbf{v}_{B'}$ (with respect to the basis B'). We call $\mathbf{C}_{B,B'}$ the **change-of-coordinates matrix**. It is the matrix that converts coordinate vectors expressed in tems of the ordered basis B to the coordinate vectors with respect to the ordered basis B'.

2.1. Calculating $\mathbf{C}_{B,B'}$.

$$\left[\left[\begin{array}{ccc} | & \cdots & | \\ \mathbf{b}'_1 & \cdots & \mathbf{b}'_n \\ | & \cdots & | \end{array}\right] \middle| \left[\begin{array}{ccc} | & \cdots & | \\ \mathbf{b}_1 & \cdots & \mathbf{b}_n \\ | & \cdots & | \end{array}\right]\right] \quad \to \quad \left[\mathbf{I} \mid \mathbf{C}_{B,B'}\right]$$

To see why this works, first recall that the sequence of row operations that reduce reduce the matrix $\mathbf{M}_{B'}$ to the identity can be also be carried out by applying a product of elementary matrices; and that the product of these elementary matrices is exactly $\mathbf{M}_{B'}^{-1}$. Visually, if

$$\mathbf{I} = \mathcal{R}_k\left(\mathcal{R}_{k-1}\left(\cdots\left(\mathcal{R}_1\left(\mathbf{M}_{B'}\right)\right)\right)\right)$$

then

$$(\mathbf{M}_{B'})^{-1} = \mathbf{E}_{\mathcal{R}_k}\mathbf{E}_{\mathcal{R}_{k-1}}\cdots\mathbf{E}_{\mathcal{R}_1}$$

and so

$$\begin{aligned}
&\mathcal{R}_k\left(\mathcal{R}_{k-1}\left(\cdots\left(\mathcal{R}_1\left(\left[\left[\begin{array}{ccc} | & \cdots & | \\ \mathbf{b}'_1 & \cdots & \mathbf{b}'_n \\ | & \cdots & | \end{array}\right] \middle| \left[\begin{array}{ccc} | & \cdots & | \\ \mathbf{b}_1 & \cdots & \mathbf{b}_n \\ | & \cdots & | \end{array}\right]\right]\right)\right)\right)\right) \\
=\;& \left[\mathbf{E}_{\mathcal{R}_k}\mathbf{E}_{\mathcal{R}_{k-1}}\cdots\mathbf{E}_{\mathcal{R}_1}\mathbf{M}_{B'} \middle| \mathbf{E}_{\mathcal{R}_k}\mathbf{E}_{\mathcal{R}_{k-1}}\cdots\mathbf{E}_{\mathcal{R}_1}\mathbf{M}_B\right] \\
=\;& \left[(\mathbf{M}_{B'})^{-1}\mathbf{M}_{B'} \mid (\mathbf{M}_{B'})^{-1}\mathbf{M}_B\right] \\
\equiv\;& \left[\mathbf{I} \mid \mathbf{C}_{B,B'}\right]
\end{aligned}$$

EXAMPLE 15.4. Calculate the **change-of-coordinates matrix** taking the basis $B = \{[1,-1],[1,1]\}$ to the basis $B' = \{[0,1],[2,1]\}$.

$$\begin{aligned}
\left[\begin{array}{cc|cc} 0 & 2 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{array}\right] \quad &\to \quad \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & 2 & 1 & 1 \end{array}\right] \\
&\to \quad \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]
\end{aligned}$$

and so

$$\mathbf{C}_{B,B'} = \left[\begin{array}{cc} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

2.2. General Change of Basis. Now let V be an arbitrary vector space (not necessarily $\mathbb{R}^n$) with basis $B = \{\mathbf{b}_1,\ldots,\mathbf{b}_n\}$, and suppose $B' = \{\mathbf{b}'_1,\ldots,\mathbf{b}'_n\}$ is another basis for V. How can we find the coordinate vector $\mathbf{v}_{B'}$ of a vector $\mathbf{v} \in V$ if we only know its coordinates with respect to the basis B?

Well, this is the same sort of problem. The coordinate vectors $\mathbf{v}_B$ and $\mathbf{v}_{B'}$ are the vectors

$$\begin{aligned}
\mathbf{v}_B &= [c_1,\ldots,c_n] \in \mathbb{R}^n \\
\mathbf{v}_{B'} &= [c'_1,\ldots,c'_n] \in \mathbb{R}^n
\end{aligned}$$

such that

$$c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = \mathbf{v} = c_1'\mathbf{b}_1' + \cdots + c_n'\mathbf{b}_n' \qquad (*)$$

Suppose we know the coordinate vectors of $\mathbf{b}_1', \ldots, \mathbf{b}_n'$ with respect to the basis B.

$$\begin{aligned}
\mathbf{b}_1' &= b_{11}'\mathbf{b}_1 + \cdots b_{1n}'\mathbf{b}_n \\
&\vdots \\
\mathbf{b}_n' &= b_{n1}'\mathbf{b}_1 + \cdots + b_{nn}'\mathbf{b}_n
\end{aligned}$$

Then, since the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are all linearly independent, the linear system (*)

$$c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = c_1'\left(b_{11}'\mathbf{b}_1 + \cdots b_{1n}'\mathbf{b}_n\right) + \cdots + c_n'\left(b_{n1}'\mathbf{b}_1 + \cdots b_{nn}'\mathbf{b}_n\right)$$

leads to

$$\begin{aligned}
c_1 &= \left(c_1'b_{11}' + \cdots + c_n'b_{n1}'\right) \\
c_2 &= \left(c_1'b_{21}' + \cdots + c_n'b_{n2}'\right) \\
&\vdots \\
c_n &= \left(c_1'b_{1n} + \cdots + c_n'b_{nn}'\right)
\end{aligned}$$

or

$$\mathbf{v}_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_{11}' & b_{n1}' \\ & \\ b_{1n}' & b_{nn'} \end{bmatrix} \begin{bmatrix} c_1' \\ \vdots \\ c_n' \end{bmatrix} = \mathbf{C}_{BB'}\mathbf{v}_{B'}$$

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