LECTURE 13

Eigenvalues and Eigenvectors

Definition 13.1. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. The* ***eigenvalue-eigenvector problem*** *for* $\mathbf{A}$ *is the problem of finding numbers* λ *and vectors* $\mathbf{v} \in \mathbb{R}^3$ *such that*

$$\mathbf{Av} = \lambda \mathbf{v} \ .$$

If λ, $\mathbf{v}$ *are solutions of a eigenvector-eigenvalue problem then the vector* $\mathbf{v}$ *is called an* ***eigenvector*** *of* $\mathbf{A}$ *and* λ *is called an* ***eigenvalue*** *of* $\mathbf{A}$.

Remark 13.2. Because eigenvectors and eigenvalues always come in pairs $(\lambda, \mathbf{v})$ one often uses language like "λ is the eigenvalue of the vector $\mathbf{v}$" or "$\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ".

Note that the multiplication on the left hand side is matrix multiplication (complicated) while the multiplication on the right hand side is scalar multiplication (easy). Thus, one way of thinking about the eigenvector-eigenvalue problem is that, when a solution is found we can simplify matrix multiplication. I remark that so doing, we'll also get a better idea of what is actually going on in many problems.

1. Motivating Examples

1.1. Rotations and Axes of Rotation. Suppose $\mathbf{R}$ is a 3×3 matrix with the property that its transpose equals its inverse:

$$\mathbf{R}^t = \mathbf{R}^{-1}$$

Such a matrix will define a linear transformation $T_R : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that preserves the lengths of vectors. For, since

$$\|\mathbf{x}\|^2 = \mathbf{x}^t\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (x_1)^2 + (x_2)^2 + (x_3)^2$$

if we set

$$\mathbf{T}_R(\mathbf{x}) = \mathbf{Rx}$$

we'll have

$$\|\mathbf{Rx}\|^2 = (\mathbf{Rx})^t(\mathbf{Rx}) = \mathbf{x}^t\mathbf{R}^t\mathbf{Rx} = \mathbf{x}^t\mathbf{R}^{-1}\mathbf{Rx} = \mathbf{x}^t\mathbf{x}$$

We call such matrices *rotation matrices*, because these are exactly the 3×3 matrices that implement rotations on 3-dimensional vectors.

Now one property of rotations is that they always have an axis of rotation: meaning there is always one line in $\mathbb{R}^3$ that is left undisturbed by the rotation. We note, for example, that in the case of the planet Earth, the line that that passes through the north and south poles is unaffected by its daily rotation.

Given an arbitary rotation matrix $\mathbf{R}$, one can thus ask: "How to figure out its axis of rotation?". Well, the condition we'd want is that

$$\mathbf{Rx} = (1)\,\mathbf{x}$$

That is, we'd want to find an eigenvector $\mathbf{x}$ of $\mathbf{R}$ with eigenvalue 1. Once we find such a vector, then we'll have the corresponding axis of rotation.

EXAMPLE 13.3. Consider the following linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$:

$$T\left([x, y, z]\right) = \left[x, y\cos(\theta) + z\sin\left(\theta\right), -y\sin(\theta) + z\cos(\theta)\right]$$

The corresponding matrix is

$$\mathbf{A}_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos\left(\theta\right) \end{bmatrix}$$

This transformation corresponds to a rotation about the x-axis by θ.

QUESTION 13.4. *What vectors in $\mathbb{R}^3$ are unaffected by this transformation?*

This question is equivalent to solving the linear system

$$\mathbf{A}_T \mathbf{x} = \mathbf{x} \tag{13.1}$$

or

$$\left(\mathbf{A}_T - \mathbf{I}\right)\mathbf{x} = \mathbf{0}$$

or

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(\theta) - 1 & \sin(\theta) \\ 0 & -\sin(\theta) & \cos\left(\theta\right) - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row-reducing the matrix on the left-hand side yields

$$\longrightarrow \begin{bmatrix} 0 & -\sin(\theta) & \cos(\theta) - 1 \\ 0 & 0 & -2\frac{\cos(\theta)-1}{\sin(\theta))} \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{c} x \text{ is arbitrary} \\ y = 0 \\ z = 0 \end{array}$$

In other words, the transformation T leaves the points along the x-axis unaffected, and **only** those points remain unaffected by the transformation.

Perhaps the preceding example is so simple-minded that the importance of the eigenvector-eigenvalue problem is obfuscated; surely a rotation about the z-axis will leave points along the z-axis fixed. However, consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that combines a rotation about z-axis with a rotation about the x-axis:

$$\begin{aligned} T \quad : \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} &\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} \left(\cos\theta\right)x + \left(\sin\theta\right)y \\ -\left(\cos\phi\sin\theta\right)x + \left(\cos\phi\cos\theta\right)y + \left(\sin\phi\right)z \\ \left(\sin\phi\sin\theta\right)x - \left(\sin\phi\cos\theta\right)y + \left(\cos\phi\right)z \end{bmatrix} \end{aligned}$$

This is in fact another rotation; but which rotation? What is the axis of rotation? The answer to this question can be found by solving

$$T\mathbf{x} = (1)\mathbf{x}$$

i.e, by solving an eigenvector-eigenvalue problem.

EXAMPLE 13.5. Let $C^\infty(\mathbb{R})$ denote the vector space of smooth (infinitely differentiable) functions on the real line. Consider the transformation $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by

$$D[f] = \frac{df}{dx}$$

It is easy to see that this is a linear transformation (it preserves scalar multiplication and vector addition in $C^\infty(\mathbb{R})$). We can associate with D an infinite matrix (employing simple monomials in x, $\left\{1, x, x^2, x^3, \ldots\right\}$

as a standard basis for $C^\infty(\mathbb{R})$):

$$\begin{aligned} D[1] &= 0 \\ D[x] &= 1 \\ D\left[x^2\right] &= 2x \\ D\left[x^3\right] &= 3x^2 \\ &\vdots \end{aligned}$$

$$\Rightarrow \qquad \mathbf{A}_D = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Solution of a differential equation of the form

$$\frac{df}{dx} = \lambda f$$

is then equivalent to a linear system of the form

$$\mathbf{A}_D \mathbf{f} = \lambda \mathbf{f} \tag{13.2}$$

Note the similarity of this equation with (13.1).

In terms of these new semantics, the first example shows that the vectors along the x-axis are eigenvectors of the linear transformation T with eigenvalues 1; and the second example shows that the solution of a differential equation can be thought of as a problem of finding an eigenvector for a linear transformation on the vector space of smooth functions on $\mathbb{R}$.

2. Computing Eigenvalues

For a given $n \times n$ matrix $\mathbf{A}$, the **eigenvalue problem** is the problem of finding the eigenvalues and eigenvectors of $\mathbf{A}$. Our abstract machinery now plays a crucial role.

Before finding a solution $\mathbf{v}$ of

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \quad , \quad \lambda \in \mathbb{R} \tag{13.3}$$

we first ask, *when do non-trivial solutions exist?*

First we rewrite (13.3) as

$$\mathbf{A}\mathbf{v} - \lambda \mathbf{v} = \mathbf{0}$$

or

$$(\mathbf{A} - \lambda \mathbf{I})\, \mathbf{v} = \mathbf{0}$$

Now recall that demanding that the existence of non-zero solutions to linear system

$$\mathbf{B}\mathbf{x} = \mathbf{0}$$

is equivalent to demanding that the null space of $\mathbf{B}$ is at least 1-dimensional, which is equivalent to saying that when $\mathbf{B}$ is row-reduced to row-echelon form there is at least one row full of zeros, which implies $\det(\mathbf{B}) = 0$.

Thus, we arrive at the following criteria for the existence of eigenvectors and eigenvalues.

ASSERTION 13.1. If non-trivial solutions of $\mathbf{Av} = \lambda \mathbf{v}$ exist then

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

EXAMPLE 13.6. Find the possible eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$

- If $\mathbf{A}$ is to have eigenvalues, then we must have

  $$0 = \det(\mathbf{A} - \lambda \mathbf{I}) = \det \begin{bmatrix} 3-\lambda & 2 \\ 2 & -\lambda \end{bmatrix} = (3-\lambda)(-\lambda) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

  Hence, the possible eigenvalues are 4 and 1.

This simple example actually serves as the prototype for the general case:

To find the eigenvalues of an $n \times n$ matrix $\mathbf{A}$:

FACT 13.7. (1) *Compute the determinant of the matrix $\mathbf{H} = \mathbf{A} - \lambda \mathbf{I}$, regarding λ as a symbolic variable.*
(2) $\det(\mathbf{A} - \boldsymbol{\lambda} \mathbf{I})$ *will be a polynomial $P_{\mathbf{A}}(\lambda)$ of degree n in λ. We shall call this polynomial the* ***characteristic polynomial*** *of $\mathbf{A}$.*
(3) *Solve $P_{\mathbf{A}}(\lambda) = 0$. The roots of this equation will be the possible eigenvalues of $\mathbf{A}$.*

At this point it is worthwhile to recall the Fundamental Theorem of Algebra.

THEOREM 13.8. *Every polynomial $p(\lambda)$ of degree n has a unique factorization in terms of linear polynomials:*

$$p(\lambda) = c(\lambda - r_1)^{m_1}(\lambda - r_2)^{m_2} \cdots (\lambda - r_s)^{m_s}$$

*where $r_1 \neq r_2 \neq \cdots \neq r_s$, and $\sum_{i=1}^{s} m_i = n$. The numbers r_i are, in general, complex numbers and are called the **roots** of $p(x)$; they coincide precisely with the solutions of $p(\lambda) = 0$. The integers m_i are called the **multiplicities** of the corresponding roots.*

This theorem tells us that the eigenvalues of a matrix can be complex numbers (in fact, this is what one must expect in general), and that they may occur with some multiplicity. The multiplicity of an eigenvalue will be very relevant to the problem of diagonalizing matrices (which we will take up in the next lecture).

EXAMPLE 13.9. Find the possible eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

- We have

  $$\det(\mathbf{A} - \lambda \mathbf{I}) = \det \begin{bmatrix} -\lambda & 2 \\ -2 & -\lambda \end{bmatrix} = \lambda^2 + 4$$

  The characteristic polynomial is thus

  $$P_{\mathbf{A}}(\lambda) = \lambda^2 + 4 = (\lambda + 2i)(\lambda - 2i)$$

  Thus, the possible eigenvalues are $\lambda = \pm 2i$.

EXAMPLE 13.10. Find the possible eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

- We have

$$\begin{aligned}
\det(\mathbf{A}-\lambda\mathbf{I}) &= \det\begin{bmatrix} -\lambda & 0 & 1 \\ -2 & 1-\lambda & 1 \\ 2 & 0 & -1-\lambda \end{bmatrix} \\
&= (-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix} - (0)\det\begin{bmatrix} -2 & 1 \\ 2 & -1-\lambda \end{bmatrix} + (1)\det\begin{bmatrix} -2 & 1-\lambda \\ 2 & 0 \end{bmatrix} \\
&= \lambda(1-\lambda)(1+\lambda) + 0 - (2)(1-\lambda) \\
&= 3\lambda - \lambda^3 - 2 \\
&= -(2+\lambda)(\lambda-1)^2
\end{aligned}$$

We thus see that $\mathbf{A}$ has two possible eigenvalues, $\lambda = -2$ and $\lambda = 1$. The eigenvalue $\lambda = 1$ occurs with multiplicity 2.

3. Computation of Eigenvectors

In finding the eigenvalues of a matrix, all we have accomplished is figuring out for what values of λ

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{13.4}$$

can have a solution. I will now show you how to find the corresponding eigenvectors $\mathbf{v}$.

This is very straight-forward given our recent discussion of homogeneous linear systems. For finding vectors $\mathbf{v}$ such that equation (13.4) is satisfied is equivalent to finding solutions of

$$(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I})\,\mathbf{x} = \mathbf{0}$$

Let's do a quick example.

Example 13.11. Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$

- In the first example, we discovered that the possible eigenvalues for this matrix are $\lambda = 4, -1$.

  Let's consider first the solution of

$$(\mathbf{A} - 4\mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{13.5}$$

(which should be an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $\lambda = 4$). We have

$$\mathbf{A} - 4\mathbf{I} = \begin{bmatrix} 3-4 & 2 \\ 2 & 0-4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$

This matrix row reduces to

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

And so the solutions of (13.5) coincide with solutions of

$$\begin{array}{c} x_1 - 2x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad x_1 = 2x_2 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad , \quad x_2 \in \mathbb{R}$$

Thus, any vector of the form

$$\mathbf{v} = r\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R}$$

will be an eigenvector of $\mathbf{A}$ with eigenvalue 4.

Now let's look for eigenvectors corresponding to the eigenvalue -1. We thus look for solutions of

$$(\mathbf{A} + \mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{13.6}$$

We have

$$\mathbf{A}+\mathbf{I}=\begin{bmatrix} 3+1 & 2 \\ 2 & 0+1 \end{bmatrix}=\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

which is row-equivalent to

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

So the solutions of (13.6) coincide with the solutions of

$$\begin{array}{c} 2x_1+x_2=0 \\ 0=0 \end{array} \quad \Rightarrow \quad x_1=-\frac{1}{2}x_2 \quad \Rightarrow \quad \mathbf{x}=x_2\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

So the corresponding eigenvectors will be vectors of the form

$$\mathbf{v}=r\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

EXAMPLE 13.12. Find the eigenvectors of

$$\mathbf{A}=\begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

- Recall from Example 15.9 that the characteristic polynomial for this matrix factorizes as

  $$P_{\mathbf{A}}(\lambda)=-(2+\lambda)(\lambda-1)^2$$

  and so we have two possible eigenvalues: $\lambda=-2$ with multiplicity 1, and $\lambda=1$ with multiplicity 2.

Let's look first for eigenvectors corresponding to the eigenvalue $\lambda=-2$. We have

$$\mathbf{A}-(-2)\mathbf{I}=\begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and so the solutions $\mathbf{x}$ of $(\mathbf{A}+2\mathbf{I})\,\mathbf{x}=\mathbf{0}$ satisfy

$$\begin{array}{c} 2x_1+x_3=0 \\ 3x_2+2x_3=0 \\ 0=0 \end{array} \quad \Rightarrow \quad \mathbf{x}=\begin{bmatrix} -\frac{1}{2}x_3 \\ -\frac{2}{3}x_3 \\ x_3 \end{bmatrix} \in span\left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right)$$

So the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue -2 are scalar multiples of the vector $\left[-2,-\frac{2}{3},1\right]$.

We'll now repeat the calculation for the double root $\lambda=1$. We have

$$\mathbf{A}-(1)\mathbf{I}=\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 1 \\ 2 & 0 & -2 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and so the solutions $\mathbf{x}$ of $(\mathbf{A}-\mathbf{I})\,\mathbf{x}=\mathbf{0}$ satisfy

$$\begin{array}{c} x_1=0 \\ -x_3=0 \\ 0=0 \end{array} \quad \Rightarrow \quad \mathbf{x}=\begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$$

So the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue 1 are scalar multiples of the vector $[0,1,0]$.

4. Eigenspaces

DEFINITION 13.13. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix and suppose* $r \in \mathbb{R}$ *is an eigenvalue of* $\mathbf{A}$*. The* r*-eigenspace of* $\mathbf{A}$ *is the subset*

$$\{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{Av} = r\mathbf{v}\}$$

LEMMA 13.14. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix and suppose* $r \in \mathbb{R}$ *is an eigenvalue of* $\mathbf{A}$*. The* r*-eigenspace of* $\mathbf{A}$ *is a subspace of* $\mathbb{R}^n$*.*

Proof. This can be seen two ways.

First of all, suppose r is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}_1, \mathbf{v}_2$ lie in r-eigenspace of $\mathbf{A}$. Then

$$\mathbf{Av}_1 = r\mathbf{v}_1 \quad \text{and} \quad \mathbf{Av}_2 = r\mathbf{v}_2$$

and so

$$\mathbf{A}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{Av}_1 + \mathbf{Av}_2 = r\mathbf{v}_1 + r\mathbf{v}_2 = r(\mathbf{v}_1 + \mathbf{v}_2)$$

Thus, $\mathbf{v}_1 + \mathbf{v}_2$ is also in the r-eigenspace of $\mathbf{A}$. We also have

$$\mathbf{A}(c\mathbf{v}_1) = c\mathbf{Av}_1 = cr\mathbf{v}_1 = r(c\mathbf{v}_1)$$

Since the r-eigenspace of $\mathbf{A}$ is closed under both vector addition and scalar multiplication, it is a subspace.

Even more easily,

$$\mathbf{v} \in r\text{-eigenspace of } \mathbf{A} \quad \Longleftrightarrow \quad \mathbf{Av} = r\mathbf{v} \quad \Longleftrightarrow \quad \mathbf{v} \text{ is a solution of } (\mathbf{A} - r\mathbf{I})\,\mathbf{x} = 0$$

So

$$r\text{-eigenspace of } \mathbf{A} = NullSp(\mathbf{A} - r\mathbf{I}) = \text{ a subspace of } \mathbb{R}^n$$

5. Two Kinds of Eigenvalue Multiplicities

Here I want to introduce some nomenclature that will become important latter.

Suppose r is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$. This means that $\lambda = r$ is a root of

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

We call the right hand side of this equation the *characterist polynomial of* $\mathbf{A}$ and we shall denote by $p_{\mathbf{A}}(\lambda)$. Thus,

$$p_{\mathbf{A}}(\lambda) \equiv \det(\mathbf{A} - \lambda\mathbf{I})$$

Since r is a root of $p_{\mathbf{A}}(\lambda) = 0$, it follows from the Fundamental Theorem of Algebra that $(\lambda - r)$ is a factor of $p_{\mathbf{A}}(\lambda)$. In fact, one has

LEMMA 13.15. *If* r *is a eigenvalue of an* $n \times n$ *matrix* $\mathbf{A}$*, there is an integer* $m \geq 1$ *such that*

$$p_{\mathbf{A}}(\lambda) = (\lambda - r)^m Q(\lambda)$$

with $Q(\lambda)$ *a polynomial of degree* $n - m$ *such that* $Q(r) \neq 0$*.*

We call the integer m in the lemma, the *algebraic multiplicity of the eigenvalue* r.

Suppose again that r is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$. The dimension of the r-eigenspace of $\mathbf{A}$ is the *geometric multiplicity of the eigenvalue* r.

Example 13.16. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

find the eigenvalues of $\mathbf{A}$ and their algebraic and geometric multiplicities.

- Well first, let's find the eigenvalues of $\mathbf{A}$. It's characteristic polynomial is

$$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{pmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

Since the matrix $\mathbf{A} - \lambda\mathbf{I}$ is already in Row Echelon Form, its determinant is just the product of its diagonal elements. Thus,

$$p_{\mathbf{A}}(\lambda) = (3-\lambda)^2 (2-\lambda)$$

From this we see that $\lambda = 3$ is an eigenvalue with algebraic multiplicity 2 and $\lambda = 2$ is an eigenvalue with algebraic multiplicity 1.

To determine the geometric muliplicities we need to determine the dimension of the corresponding eigenspaces.

- $\lambda = 3$ eigenspace.

The 3-eigespace is the null space of

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Row-reducing this matrix (pretty easily) we get

$$\begin{aligned} \text{3-eigenspace} &= NullSp \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \text{solutions set of } \begin{cases} x_2 = 0 \\ x_3 = 0 \end{cases} \\ &= span\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Since we have one basis vector the dimension of the 3-eigenspace is 1. Thus, the geometric multiplicity of $\lambda = 3$ is 1. (Recall its algebraic multiplicity was 2.)

- $\lambda = 2$ eigenspace

The 2-eigenspace is the solution set of $(\mathbf{A} - 2\mathbf{I})\,\mathbf{x} = \mathbf{0}$. We have

$$(\mathbf{A} - 2\mathbf{I}) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{row reduces to}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so the null space is corresponds to solutions of

$$\begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in span\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Since we have one basis vector, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, the 2-eigenspace is 1-dimensional. Hence, the geometric multiplicity of the eigenvalue $\lambda = 2$ is 1.

Remark 13.17. Actually, to determine the geometric multiplicity of an eigenvalue r all we really had to do is row-reduce $\mathbf{A} - r\mathbf{I}$ to row echelong form and count the number of columns without pivots - that would give the dimension of $NullSp(\mathbf{A} - r\mathbf{I})$ which is the same thing is the geometric multiplicity of the eigenvalue r.