LECTURE 11

Review Session for Second Midterm

I. Formal Definitions

  A. *Dimension*

  The *dimension* of a subspace W is the number of vectors in any basis for W.

  B. *Row Space*

  The **row space** of an $n \times m$ matrix $\mathbf{A}$ is subspace of $\mathbb{R}^m$ corresponding to the span of the row vectors of $\mathbf{A}$.

  C. *Column Space*

  The **column space** of an $n \times m$ matrix $\mathbf{A}$ is the subspace of $\mathbb{R}^n$ corresponding to the span of the column vectors of $\mathbf{A}$.

  D. *Null Space*

  The **null space** of an $n \times m$ matrix $\mathbf{A}$ is the solution set of the linear system $\mathbf{Ax} = \mathbf{0}_{\mathbb{R}^m}$

  E. *Rank*

  The **rank** of an $n \times m$ matrx $\mathbf{A}$ is the common dimension of its row and column spaces.

  F. *Linear Transformation:*

  A **linear transformation** is a function $T : \mathbb{R}^m \to \mathbb{R}^n$ such that

  (i) $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$.

  (ii) $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$

  G. *Range*

  The **range** of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is the following subset of the codomain $\mathbb{R}^n$

$$Range(T) = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$$

  H. *Kernel*

  The **kernel** of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is the following subset of the domain $\mathbb{R}^m$

$$Ker(T) = \{\mathbf{x} \in \mathbb{R}^m \mid T(\mathbf{x}) = \mathbf{0}_{\mathbb{R}^n}\}$$

II. Using row reduction to identify bases for subspaces

III. Working with Linear Transformations

  A. Proving a subset of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

  B. Constructing the $n \times m$ matrix attached to a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$

  C. Finding the range and kernel of a linear transformation

IV. Determinants

  A. Calculating Determinants using cofactor expansions

  B. Calculating Determinants using row reduction

  C. Solving square linear systems via Crammer's Rule

  D. Inverting square matrices using cofactors

Math 3013
SAMPLE SECOND EXAM

1. Write down the formal definitions of the following notions:

(a) a *linear transformation from* $\mathbb{R}^m$ to $\mathbb{R}^n$

(b) the *range* of a linear transfomation $T : \mathbb{R}^m \to \mathbb{R}^n$

(c) the *kernel* of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$

2. Consider the following mapping: $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_2, x_1 - x_3]$. Show that T is a linear transformation.

3. Suppose T is the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ given by

$$T([x_1, x_2, x_3]) = [x_1 + x_2 \ , \ -x_1 + x_3 \ , \ x_2 + x_3 \ , \ 0]$$

(a) Find the matrix $\mathbf{A}_T$ such that $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

(b) Find a basis for the *range* of T

(c) Find a basis for the *kernel* of T.

4. Compute the following determinants by the indicated method

(a) $\det \begin{pmatrix} 0 & 4 & -3 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ via row reduction

(b) $\det \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ via a cofactor expansion

5. Use Crammer's Rule to solve the following linear system.

$$\begin{aligned} 2x_1 + x_2 &= 5 \\ x_1 - x_2 &= -2 \end{aligned}$$

6. Find the cofactor matrix of the following matrix $\mathbf{A}$ and then use the cofactor matrix to compute $\mathbf{A}^{-1}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

Math 3013
SOLUTIONS TO SAMPLE SECOND EXAM

1. Write down the formal definitions of the following notions:

(a) a *linear transformation from* $\mathbb{R}^m$ to $\mathbb{R}^n$

- A linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ is a function $T : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\begin{aligned} T(\mathbf{x}_1 + \mathbf{x}_2) &= T(\mathbf{x}_1) + T(\mathbf{x}_2) && \text{for all } \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m \\ T(\lambda \mathbf{x}) &= \lambda T(\mathbf{x}) && \text{for all } \lambda \in \mathbb{R} \text{ and all } \mathbf{x} \in \mathbb{R}^m \end{aligned}$$

(b) the *range* of a linear transfomation $T : \mathbb{R}^m \to \mathbb{R}^n$

- The range of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is the set

$$Range(T) = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^m\} \subset \mathbb{R}^n$$

(c) the *kernel* of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$

- The kernel of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is the set

$$Ker(T) = \{\mathbf{x} \in \mathbb{R}^m \mid T(\mathbf{x}) = \mathbf{0}\} \subset \mathbb{R}^m$$

2. Consider the following mapping: $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_2, x_1 - x_3]$. Show that T is a linear transformation.

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$$T(\lambda [x_1, x_2, x_3]) = T([\lambda x_1, \lambda x_2, \lambda x_3]) = [\lambda x_2, \lambda x_1 - \lambda x_3] = \lambda [x_2, x_1 - x_3] = \lambda T([x_1, x_2, x_3]) \quad \Rightarrow \quad T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$$

$$T(\mathbf{x} + \mathbf{x}') = T([x_1 + x_1', x_2 + x_2', x_3 + x_3']) = [x_2 + x_2', (x_1 + x_1') - (x_3 + x_3')] = [x_2, x_1 - x_3] + [x_2', x_1' - x_3'] = T(\mathbf{x}) + T(\mathbf{x}')$$

Since T preserves scalar multiplication and vector addition, T is a linear transformation. $\square$

3. Suppose T is the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ given by

$$T([x_1, x_2, x_3]) = [x_1 + x_2 \ , \ -x_1 + x_3 \ , \ x_2 + x_3 \ , \ 0]$$

(a) Find the matrix $\mathbf{A}_T$ such that $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

- We first calculate the action of T on the standard basis vectors for the domain $\mathbb{R}^3$:

$$\begin{aligned} T(\mathbf{e}_1) &= T([1,0,0]) = [1,-1,0,0] \\ T(\mathbf{e}_2) &= T([0,1,0]) = [1,0,1,0] \\ T(\mathbf{e}_3) &= T([0,0,1]) = [0,1,1,0] \end{aligned}$$

Converting these to columns gives us the matrix $\mathbf{A}_T$

$$\mathbf{A}_T = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) Find a basis for the *range* of T

- The range of T is equivalent to the column space of $\mathbf{A}_T$. To find the latter we first row reduce $\mathbf{A}_T$ to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{row reduction}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since we only have pivots in the first two columns of the row echelon form, the first two columns of $\mathbf{A}_T$ will provide a basis for the column space of $\mathbf{A}_T$, and so also (once reinterpreted as vectors in $\mathbb{R}^4$) a basis for the range of T :

$$\text{basis for } ColSp\left(\mathbf{A}_T\right) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \Rightarrow \quad \text{basis for } Range\left(T\right) = \{[1,-1,0,0]\ ,\ [1,0,1,0]\}$$

(c) Find a basis for the *kernel* of T.

- The kernel of T will correspond to the null space of the matrix $\mathbf{A}_T$ (i.e., the solution set of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$). Since we have already row reduced $\mathbf{A}_T$ to a reduced row echelon form in part (b) above, we can use that RREF for $\mathbf{A}_T$ to determine a basis for the null space of $\mathbf{A}_T$:

$$NullSp\left(\mathbf{A}_T\right) = NullSp \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{solution set of} \left. \begin{array}{c} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{array} \right\}$$

Since the third column of the RREF does not contain a pivot, x_3 is to be regarded as a free parameter. Writing the general solution vector in terms of the free parameter we get

$$\mathbf{x} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

We can now conclude

$$\text{basis for } NullSp\left(\mathbf{A}_t\right) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} \quad \Rightarrow \quad \text{basis for } Ker\left(T\right) = \{[1,-1,1]\}$$

4. Compute the following determinants by the indicated method

(a) $\det \begin{pmatrix} 0 & 4 & -3 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ via row reduction

- We have

$$\begin{aligned} \det \begin{pmatrix} 0 & 4 & -3 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} &\xrightarrow{R_3 \longleftrightarrow R_1} -\det \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 4 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\xrightarrow{R_3 \to R_3 - 2R_2} -\det \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= -\left(1\right)\left(2\right)\left(-3\right)\left(1\right) \\ &= 6 \end{aligned}$$

(the sign flip because we interchanged rows)

(b) $\det\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ via a cofactor expansion

- Cofactor expansion along the second row:

$$\begin{aligned}
\det\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} &= (0)(-1)^{2+1}\det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} + (0)(-1)^{2+1}\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + (2)(-1)^{2+3}\det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
&= 0 + 0 + (-2)(2) \\
&= 4
\end{aligned}$$

5. Use Crammer's Rule to solve the following linear system.

$$\begin{aligned}
2x_1 + x_2 &= 5 \\
x_1 - x_2 &= -2
\end{aligned}$$

- Casting this 2×2 linear system in the form $\mathbf{Ax} = \mathbf{b}$, we have

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \quad , \quad \mathbf{b} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

and

$$\mathbf{B}_1 = \begin{pmatrix} 5 & 1 \\ -2 & -1 \end{pmatrix} \quad , \quad \mathbf{B}_2 = \begin{pmatrix} 2 & 5 \\ 1 & -2 \end{pmatrix}$$

Crammer's Rule says that the components x_1, x_2 of the solution vector are given by

$$x_i = \frac{\det(\mathbf{B}_i)}{\det(\mathbf{A})} \quad , \quad i = 1, 2$$

Now

$$\begin{aligned}
\det(\mathbf{A}) &= (2)(-1) - (1)(1) = -3 \\
\det(\mathbf{B}_1) &= (5)(-1) - (1)(-2) = -3 \\
\det(\mathbf{B}_2) &= (2)(-2) - (5)(1) = -9
\end{aligned}$$

and so

$$\begin{aligned}
x_1 &= \frac{-3}{-3} = 1 \\
x_2 &= \frac{-9}{-3} = 3
\end{aligned}$$

6. Find the cofactor matrix of the following matrix $\mathbf{A}$ and then use the cofactor matrix to compute $\mathbf{A}^{-1}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

- We first note that (by a cofactor expansion along the second row of $\mathbf{A}$)

$$\det(\mathbf{A}) = 0 + (1)(-1)^{2+2}\det\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} + 0 = 1$$

The entries of the cofactor matrix of $\mathbf{A}$ are given by

$$c_{ij} = (-1)^{i+j}\det(\mathbf{A}_{ij})$$

where $\mathbf{A}_{ij}$ is the $(ij)^{th}$ minor of $\mathbf{A}$. Thus,

$$\begin{aligned}
c_{11} &= (-1)^{1+1}\det\begin{pmatrix}1&0\\0&3\end{pmatrix}=3 \quad , \quad c_{12}=(-1)^{1+2}\det\begin{pmatrix}0&0\\1&3\end{pmatrix}=0 \quad , \quad c_{13}=(-1)^{1+3}\det\begin{pmatrix}0&1\\1&0\end{pmatrix}=-1\\
c_{21} &= (-1)^{2+1}\det\begin{pmatrix}0&2\\0&3\end{pmatrix}=0 \quad , \quad c_{22}=(-1)^{2+2}\det\begin{pmatrix}1&2\\1&3\end{pmatrix}=1 \quad , \quad c_{23}=(-1)^{2+3}\det\begin{pmatrix}1&0\\1&0\end{pmatrix}=0\\
c_{31} &= (-1)^{3+1}\det\begin{pmatrix}0&2\\1&0\end{pmatrix}=-2 \quad , \quad c_{32}=(-1)^{3+2}\det\begin{pmatrix}1&2\\0&0\end{pmatrix}=0 \quad , \quad c_{33}=(-1)^{3+3}\det\begin{pmatrix}1&0\\0&1\end{pmatrix}=1
\end{aligned}$$

Thus,

$$\mathbf{C}=\begin{pmatrix}3&0&-1\\0&1&0\\-2&0&1\end{pmatrix}$$

and

$$\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\mathbf{C}^T=\frac{1}{1}\begin{pmatrix}3&0&-2\\0&1&0\\-1&0&1\end{pmatrix}=\begin{pmatrix}3&0&-2\\0&1&0\\-1&0&1\end{pmatrix}$$