LECTURE 7

Review Session for First Midterm

I. Vectors in $\mathbb{R}^n$
   A. Vector Addition
   B. Scalar Multiplication
   C. Linear Combinations of Vectors
   D. Dot Products
II. Geometry of Vector Spaces
   A. Lines, Planes and Hyperplanes
   B. Solutions Sets of Linear Equations
III. Matrices and Matrix Algebra
   A. Matrices and Linear Systems : Augmented Matrices
   B. Matrix Multiplication
   C. Matrix Addition
   D. The Transpose of a Matrix
IV. Systems of Linear Equations
   A. Linear Systems and Matrices
   B. Elementary Row Operations
   C. Row-Echelon Form
   D. Reduced Row-Echelon Form
   E. Solving Linear Systems
V. Inverses of Square Matrices
   A. Definition and Properties of Matrix Inverses
   B. Elementary Matrices
   C. Calculating Matrix Inverses
   D. Matrix inverses and $n \times n$ linear systems
VI. Subspaces and Bases
   A. Definition of Subspace
      (i) Proving that a set is a subspace
      (ii) Showing that a set is not a subspace
   B. Prototypical Subspaces
      (i) span of a set of vectors
      (ii) solution set of homogeneous linear system
   C. Bases and Linear Independence
      (i) Definition of Bases
      (ii) Definition of Linear Independence
      (iii) Finding a basis for the solution set of a homogeneous linear system
      (iv) Determining if a set of vectors is a basis for its span / finding such a basis
   D. Subspaces Attached to a Matrix and their Bases
      (i) $RowSp(\mathbf{A})$
      (ii) $ColSp(\mathbf{A})$
      (iii) $NullSp(\mathbf{A})$

Math 3013
SAMPLE FIRST EXAM

1. Let

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Compute the matrix product **BA**

2. For each of the following augmented matrices, indicate

- the number of equations and the number of variables in the corresponding linear system
- whether or not the corresponding linear system has a solution
- if the corresponding linear system does have a solution, the number of free variables in the solution.

Hint: note that these augmented matrices are already in row echelon form.

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

3. (10 pts) Consider the following linear system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= -1 \\ -x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

Write down the corresponding augmented matrix and row reduce it to row-echelon form.

4. Row reduce the following augmented matrix to **reduced** row-echelon form.

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccccc|c} 2 & 2 & 4 & 6 & 2 & 2 \\ 0 & 0 & 3 & 6 & 6 & 3 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

5. Suppose the augmented matrix below is the Reduced Row Echelon Form of an augmented matrix of a linear system. Display the solution of the linear system as a hyperplane (within the space of variables).

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

6. (10 pts) Compute the inverse of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

7. Write down precise definitions for the following terms

(a) **Subspace of** $\mathbb{R}^n$:

(b) **Basis of a Subspace**:

(c) **Linearly Independent Set of Vectors**:

8. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \end{pmatrix}$$

Find bases for

(a) the row space, $RowSp(\mathbf{A})$, of $\mathbf{A}$

(b) the column space, $ColSp(\mathbf{A})$, of $\mathbf{A}$

(c) the null space, $NullSp(\mathbf{A})$, of $\mathbf{A}$

Math 3013
SOLUTIONS TO SAMPLE FIRST EXAM

1. Let

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Compute the matrix product **BA**

$$\mathbf{BC} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & -1 & 3 \end{bmatrix}$$

:

2. For each of the following augmented matrices, indicate

- the number of equations and the number of variables in the corresponding linear system
- whether or not the corresponding linear system has a solution
- if the corresponding linear system does have a solution, the number of free variables in the solution.

Hint: note that these augmented matrices are already in row echelon form.

(a) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- This augmented matrix comes from a system of 4 equations and 4 unknowns. There is a solution. Since there is one column without a pivot, there is exacly one free variable (x_3) in the solution.

(b) $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$

- This augmented matrix comes from a system of 3 equations in 4 unknowns. There is no solution since the third row correspond to the equation $0 = -1$.

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

- This augmented matrix comes from a system of 4 equations in 3 unknowns. There is a solution. Since there are no columns without pivots, there are no free parameters in the solution. The solution is therefore unique.

3. (10 pts) Consider the following linear system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= -1 \\ -x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

Write down the corresponding augmented matrix and row reduce it to row-echelon form.

- The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & -1 \\ -1 & 1 & 2 & 3 \end{array}\right]$$

  Carrying out the row reduction

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & -1 \\ -1 & 1 & 2 & 3 \end{array}\right] \xrightarrow[]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

  This last matrix is in row echelon form.

4. Row reduce the following augmented matrix to **reduced** row-echelon form.

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccccc|c} 2 & 2 & 4 & 6 & 2 & 2 \\ 0 & 0 & 3 & 6 & 6 & 3 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- $$\left[\begin{array}{ccccc|c} 2 & 2 & 4 & 6 & 2 & 2 \\ 0 & 0 & 3 & 6 & 6 & 3 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[]{\substack{R_1 \to \frac{1}{2}R_1 \\ R_2 \to \frac{1}{3}R_2 \\ R_3 \to -\frac{1}{2}R_3}} \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[]{\substack{R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 2R_3}} \left[\begin{array}{ccccc|c} 1 & 1 & 2 & 3 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

  This last matrix is in reduced row echelon form

5. Suppose the augmented matrix below is the Reduced Row Echelon Form of an augmented matrix of a linear system. Display the solution of the linear system as a hyperplane (within the space of variables).

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- The equations corresponding to this augmented matrix are

$$\begin{aligned} x_2 - 2x_4 + x_5 &= 1 \\ x_3 + x_4 - x_5 &= 2 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Since columns 1, 4, and 5 do not contain pivots, x_1, x_4 and x_5 should be interpreted as *free variables* in the solution. The above equations then allow us to express x_2 and x_3 in terms of the free variables:

$$\begin{aligned} x_2 &= 1 + 2x_4 - x_5 \\ x_3 &= 2 - x_4 + x_5 \end{aligned}$$

Thus, a typical solution vector would be

$$\mathbf{x} = \begin{bmatrix} x_1 \\ 1 + 2x_4 - x_5 \\ 2 - x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The expression on the right exhibits the solutions as the elements of a hyperplane.

6. (10 pts) Compute the inverse of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

- We form the matrix $[\mathbf{A}|\mathbf{I}]$ as

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

This matrix row reduces to the following matrix in reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -6 & 3 & 1 \\ 0 & 1 & 0 & 5 & -2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{bmatrix} = \left[\mathbf{I}|\mathbf{A}^{-1}\right]$$

and so

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -6 & 3 & 1 \\ 0 & 1 & 0 & 5 & -2 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{bmatrix}$$

7. Write down precise definitions for the following terms

(a) **Subspace of** $\mathbb{R}^n$:

- A **subspace of** $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ such that
  - if $\mathbf{v}, \mathbf{w} \in S$, then so is $\mathbf{v} + \mathbf{w}$
  - if $\lambda \in \mathbb{R}$ and $\mathbf{v} \in S$, then so is $\lambda \mathbf{v}$

(b) **Basis of a Subspace**:

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for a subspace S if every vector $\mathbf{w} \in S$ can be written
$$\mathbf{w} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k$$
for one and only one choice of coefficients $c_1, \ldots, c_k$.

(c) **Linearly Independent Set of Vectors**:

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent if the only solution of
$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_k \mathbf{v}_k = \mathbf{0}$$
is
$$x_1 = 0 \ , \ x_2 = 0 \ , \ \ldots \ , \ x_k = 0$$

8. Consider the matrix
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 \end{pmatrix}$$
, row echelon form: Find bases for

(a) the row space, $RowSp(\mathbf{A})$, of $\mathbf{A}$

(b) the column space, $ColSp(\mathbf{A})$, of $\mathbf{A}$

(c) the null space, $NullSp(\mathbf{A})$, of $\mathbf{A}$

- The matrix $\mathbf{A}$ readily reduces the following reduced row echelon form
$$\mathbf{A}' = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
- As a basis for $RowSp(\mathbf{A})$ we can use the non-zero rows of any row echelon form of $\mathbf{A}$, thus
$$\text{basis for row space} = \{[1, 0, -1, -1], [0, 1, 1, 1]\}$$
- To get a basis for $ColSp(\mathbf{A})$, we note the pivots in the row echelon form $\mathbf{A}'$ occur in columns 1 and 2. We can therefore use columns 1 and 2 for the original matrix $\mathbf{A}$ as a basis form the column space of $\mathbf{A}$
$$\text{basis for column space} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$
- To get a basis for the null space of $\mathbf{A}$ we need to solve the linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$. Reducing $[\mathbf{A} \mid \mathbf{0}]$ to reduced row echelon form yields
$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding equations are

$$\begin{aligned} x_1 - x_3 - x_4 &= 0 \\ x_2 + x_3 + x_4 &= 0 \\ 0 &= 0 \end{aligned}$$

Since columns 3 and 4 of the augmented matrix in reduced row echelon form do not contain pivots, x_3 and x_4 will be the free parameters in the solution. We have

$$\begin{aligned} x_1 &= x_3 + x_4 \\ x_2 &= -x_3 - x_4 \end{aligned}$$

and so a typical solution vector will be

$$\mathbf{x} = \begin{bmatrix} x_3 + x_4 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, a basis for the null space of $\mathbf{A}$ will be

$$\text{basis for null space of } \mathbf{A} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$