LECTURE 6

Inverses of Square Matrices

1. Introduction

To motivate our discussion of matrix inverses, let me recall the solution of a linear equation in one variable:

$$ax = b \tag{6.1}$$

This is achieved simply by multiplying both sides by a^{-1}. Put another way, in more formal language, to solve (6.1) we multiply both sides by the multiplicative inverse of a.

In the preceding lectures we have seen that, by adopting a matrix formulation, we can rewrite a linear system consisting of m equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{6.2}$$

as a matrix equation

$$\mathbf{Ax} = \mathbf{b} \tag{6.3}$$

which, at least notationally, has the same form as (6.1). In fact, as a result of our fore-sighted choice of matrix notation, we can actually solve (6.3) in the same manner as we solved (6.1) whenever we can find a multiplicative inverse $\mathbf{A}^{-1}$ of the matrix.

However, before we try to push this analogy too far, let me point out its limitations. In the case of real numbers, every number except 0 has a multiplicative inverse; however, it is not true that every non-zero matrix has an inverse. In fact, in general matrices **do not** have inverses. For if $\mathbf{A}$ is an $m \times n$ matrix then we cannot have a $r \times s$ matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$$

unless $m = n = r = s$ (otherwise, one of the products $\mathbf{A}^{-1}\mathbf{A}$ or $\mathbf{A}\mathbf{A}^{-1}$ is not defined). And even when we restrict attention to square matrices (i.e. $n \times n$ matrices), we can find non-zero matrices that do not have inverses. For example, to find an inverse of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

we would need to find a matrix

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix}$$

Looking at the entries in the first column of the first row on both sides we see that this requires in particular that $1 = 0$; an obvious contradiction.

2. Properties of Matrix Inverses

Before we actually learn how to compute matrix inverses, I shall outline some of their elementary properties.

DEFINITION 6.1. *An $n \times n$ matrix $\mathbf{A}$ is **invertible** if there exists an $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$, the $n \times n$ identity matrix. Such a matrix $\mathbf{C}$ is called an **inverse** of $\mathbf{A}$. If an $n \times n$ matrix $\mathbf{A}$ is not invertible, it is called **singular**.*

THEOREM 6.2. *If an $n \times n$ matrix is invertible, then its inverse is unique.*

Proof. Let $\mathbf{C}$ and $\mathbf{D}$ be matrices such that $\mathbf{AC} = \mathbf{I}$ and $\mathbf{DA} = \mathbf{I}$. Then, one the one hand, we have

$$\mathbf{D}(\mathbf{AC}) = (\mathbf{DA})\,\mathbf{C} = (\mathbf{I})\mathbf{C} = \mathbf{C}$$

and, on the other,

$$\mathbf{D}(\mathbf{AC}) = \mathbf{D}(\mathbf{I}) = \mathbf{D}$$

and so

$$\mathbf{C} = \mathbf{D}$$

NOTATION 6.3. Henceforth we shall denote the unique inverse of an $n \times n$ matrix $\mathbf{A}$ by $\mathbf{A}^{-1}$.

THEOREM 6.4. *Let $\mathbf{A}$ and $\mathbf{B}$ be invertible $n \times n$ matrices. Then their product $\mathbf{AB}$ is also invertible and*

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Proof. A direct computation shows that

$$\begin{aligned}
\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)(\mathbf{AB}) &= \mathbf{B}^{-1}\left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{B} \\
&= \mathbf{B}^{-1}(\mathbf{I})\,\mathbf{B} \\
&= \mathbf{B}^{-1}(\mathbf{IB}) \\
&= \mathbf{B}^{-1}\mathbf{B} \\
&= \mathbf{I}
\end{aligned}$$

and similarly

$$(\mathbf{AB})\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) = \mathbf{I}$$

Since matrix inverses are unique, we can conclude that $\mathbf{AB}$ is invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

3. Elementary Matrices

Before I say more about matrix inverses, let me introduce an important auxiliary idea: that of *elementary matrices.*

DEFINITION 6.5. *Suppose $\mathcal{R}$ is an elmentary row operation (acting on $n \times n$ matrices). The **elementary matrix** $\mathbf{E}_{\mathcal{R}}$ corresponding to $\mathcal{R}$ is the $n \times n$ matrix obtained by applying $\mathcal{R}$ to the $n \times n$ identity matrix $\mathbf{I}_n$. :*

$$\mathbf{E}_{\mathcal{R}} \equiv \mathcal{R}(\mathbf{I}_n)$$

Here are some examples of elementary matrices (mad from the the 2×2 identity matrix).

(1) Suppose $\mathcal{R}_{R_1 \longleftrightarrow R_2}$ is the operation that interchanges the first and second row of a matrix. Then

$$\mathbf{E}_{\mathcal{R}_{R_1 \longleftrightarrow R_2}} = \mathcal{R}_{R_1 \longleftrightarrow R_2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(2) Suppose $\mathcal{R}_{R_2 \to \lambda R_2}$, is the operation that replaces the second row of a matrix with its scalar multiple by λ. Then

$$\mathbf{E}_{\mathcal{R}_{R_2\to\lambda R_2}} = \mathcal{R}_{R_2\to\lambda R_2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

(3) Suppose $\mathcal{R}_{R_2 \to R_2+\lambda R_1}$ is the operation replaces the second row of a matrix with its sum with λ times the second row of the matrix. Then

$$\mathbf{E}_{\mathcal{R}_{R_2\to R_2+\lambda R_1}} = \mathcal{R}_{R_2\to R_2+\lambda R_1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

THEOREM 6.6. *Suppose* $\mathbf{A}$ *is an* $n \times m$ *matrix and* $\mathbf{E}_{\mathcal{R}}$ *is the elementary matrix obtained by applying an elementary row operation to* $\mathbf{I}_n$. *Then*

$$\mathcal{R}(\mathbf{A}) = \mathbf{E}_{\mathcal{R}}\mathbf{A}$$

(i.e, the effect of the elementary row operation $\mathcal{R}$ *on* $\mathbf{A}$ *is the same as multiplying* $\mathbf{A}$ *from the left by the elementary matrix* $\mathbf{E}_{\mathcal{R}}$*).*

Rather than prove this theorem, let me just demonstrate how this works for the three examples given above. Let $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$.

(1) We have

$$\mathcal{R}_{R_1\longleftrightarrow R_2}(\mathbf{A}) = \mathcal{R}_{R_1\longleftrightarrow R_2}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} d & e & f \\ a & b & c \end{pmatrix}$$

and

$$\mathbf{E}_{\mathcal{R}_{R_1\longleftrightarrow R_2}}\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 0+d & 0+e & 0+f \\ a+0 & b+0 & c+0 \end{pmatrix} = \begin{pmatrix} d & e & f \\ a & b & c \end{pmatrix}$$

and so

$$\mathcal{R}_{R_1\longleftrightarrow R_2}(\mathbf{A}) = \mathbf{E}_{\mathcal{R}_{R_1\longleftrightarrow R_2}}\mathbf{A}$$

(2) We have

$$\mathcal{R}_{R_2\to\lambda R_2}(\mathbf{A}) = \mathcal{R}_{R_2\to\lambda R_2}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a & b & c \\ \lambda d & \lambda e & \lambda f \end{pmatrix}$$

and

$$\mathbf{E}_{\mathcal{R}_{R_2\to\lambda R_2}}\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a+0 & b+0 & c+0 \\ 0+\lambda d & 0+\lambda e & 0+\lambda f \end{pmatrix} = \begin{pmatrix} a & b & c \\ \lambda d & \lambda e & \lambda f \end{pmatrix}$$

and so

$$\mathcal{R}_{R_2\to\lambda R_2}(\mathbf{A}) = \mathbf{E}_{\mathcal{R}_{R_2\to\lambda R_2}}\mathbf{A}$$

(3) We have

$$\mathcal{R}_{R_2\to R_2+\lambda R_1}(\mathbf{A}) = \mathcal{R}_{R_2\to R_2+\lambda R_1}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a & b & c \\ d+\lambda a & e+\lambda b & f+\lambda c \end{pmatrix}$$

and

$$\mathbf{E}_{\mathcal{R}_{R_2\to R_2+\lambda R_1}}\mathbf{A} = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} a+0 & b+0 & c+0 \\ \lambda a+d & \lambda b+e & \lambda c+f \end{pmatrix}$$

and so

$$\mathcal{R}_{R_2\to R_2+\lambda R_1}(\mathbf{A}) = \mathbf{E}_{\mathcal{R}_{R_2\to R_2+\lambda R_1}}\mathbf{A}$$

LEMMA 6.7. *Every elementary matrix has an inverse, which is also an elementary matrix.*

Proof. Recall that an elementary matrix is a matrix that is obtained from an identity matrix by a single elementary row operation.

Let $\mathbf{E}$ be the elementary matrix corresponding to the row operation that exchanges the i^{th} and j^{th} rows. Then if exchange the the i^{th} and j^{th} rows again, we get back where we started; in other words we must have

$$\mathbf{EE} = \mathbf{I}$$

so $\mathbf{E}$ is its own inverse.

Now suppose $\mathbf{E}$ is the elementary matrix corresponding to the rescaling of a particular row by a factor $k \neq 0$, and $\mathbf{E}'$ is the elementary matrix corresponding to rescaling that same row by a factor k^{-1}. Then we have

$$\mathbf{EE}' = \mathbf{I}$$

So elementary matrices corresponding to rescalings have inverses as well.

Finally let $\mathbf{E}$ be the elementary matrix corresponding to replacing the j^{th} row with its sum with k times the i^{th} row. This operation can be undone by replacing the j^{th} row by its sum with $-k$ times the i^{th} row. Let $\mathbf{E}'$ be the elementary matrix corresponding to this latter row operation. Then we have

$$\mathbf{EE}' = \mathbf{I}$$

so elementary matrices corresponding to the replacements of rows by their sums with multiples of other rows have inverses (which are themselves elementary matrices).

Thus, every elementary matrix has an inverse and that inverse is an elementary matrix.

4. The Fundamental Theorem of Invertible Matrices

LEMMA 6.8. *Let $\mathbf{A}$ be an $n \times n$ matrix. The following statements are equivalent (if one statement is true for $\mathbf{A}$ then all these statements are true for $\mathbf{A}$).*

(a) $\mathbf{A}$ *is invertible.*
(b) $\mathbf{Ax} = \mathbf{b}$ *has a unique solution for each vector* $\mathbf{b} \in \mathbb{R}^n$.
(c) $\mathbf{Ax} = \mathbf{0}$ *has only the trivial soluion* $\mathbf{x} = \mathbf{0}$.
(d) *The reduced row echelon form of* $\mathbf{A}$ *is* $\mathbf{I}_n$ *(the $n \times n$ identity matrix).*
(e) $\mathbf{A}$ *is a product of elementary matrices.*

Proof. We'll demonstrate the following chain of implications

$$(a) \quad \Rightarrow \quad (b) \quad \Rightarrow \quad (c) \quad \Rightarrow \quad (d) \quad \Rightarrow \quad (e) \quad \Rightarrow \quad (a)$$

$(a) \quad \Rightarrow \quad (b)$: Suppose $\mathbf{A}$ is invertible, with inverse $\mathbf{A}^{-1}$. I claim $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is a solution of $\mathbf{Ax} = \mathbf{b}$. Indeed,

$$\mathbf{A}\left(\mathbf{A}^{-1}\mathbf{b}\right) = \left(\mathbf{AA}^{-1}\right)\mathbf{b} = \mathbf{I}_n\mathbf{b} = \mathbf{b}$$

and so $\mathbf{A}^{-1}\mathbf{b}$ is a solution. Suppose $\mathbf{y}$ is another solution of $\mathbf{Ax} = \mathbf{b}$. Then

$$\mathbf{Ay} = \mathbf{b} \quad \Rightarrow \quad \mathbf{A}^{-1}\mathbf{Ay} = \mathbf{A}^{-1}\mathbf{b} \quad \Rightarrow \quad \mathbf{I}_n\mathbf{y} = \mathbf{A}^{-1}\mathbf{b} \quad \Rightarrow \quad \mathbf{y} = \mathbf{A}^{-1}\mathbf{b}$$

(so $\mathbf{y}$ is the same solution.)

$(b) \quad \Rightarrow \quad (c)$: (c) follows from (b) by simply choosing $\mathbf{b} = \mathbf{0}$, and noting that, in this case, the unique solution stipulated by (b) is

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$$

$(c) \quad \Rightarrow \quad (d)$: Assume (c), then the unique solution to $\mathbf{Ax} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{which corresponds to the augmented matrix} \quad \left[\begin{array}{cccccc|c} 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{array}\right]$$

for which the coefficient part (the "$\mathbf{A}$-part" the augmented matrix) is the identity matrix $\mathbf{I}_n$. On the other hand, since all solutions to a linear system $\mathbf{Ax} = \mathbf{0}$ are obtainable by row reducing the augmented matrix $[\mathbf{A} \mid \mathbf{b}]$ to its Reduced Row Echelon Form, we can conclude that $\mathbf{A}$ must be row reducible to the identity matrix.

$(d) \quad \Rightarrow \quad (e)$: Assume $\mathbf{A}$ can be row reduced to the identity matrix, there must be a sequence $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$ of elementary row operations that systematically converts $\mathbf{A}$ to the identity matrix $\mathbf{I}_n$. Say,

$$\mathcal{R}_k \left(\mathcal{R}_{k-1} \left(\cdots \left(\mathcal{R}_1 \left(\mathbf{A} \right) \right) \right) \right) = \mathbf{I}_n$$

However, we could also implement these same elementarty row operations by left multiplication by the corresponding elementary matrices:

$$\mathbf{E}_{\mathcal{R}_k} \mathbf{E}_{\mathcal{R}_{k-1}} \cdots \mathbf{E}_{\mathcal{R}_1} \mathbf{A} = \mathcal{R}_k \left(\mathcal{R}_{k-1} \left(\cdots \left(\mathcal{R}_1 \left(\mathbf{A} \right) \right) \right) \right) = \mathbf{I}_n$$

Since each elementary matrix is invertible, we can multiply this equation above by $\mathbf{E}_{\mathcal{R}_1}^{-1} \mathbf{E}_{\mathcal{R}_2}^{-1} \cdots \mathbf{E}_{\mathcal{R}_k}^{-1}$ to get

$$\mathbf{E}_{\mathcal{R}_1}^{-1} \mathbf{E}_{\mathcal{R}_2}^{-1} \cdots \mathbf{E}_{\mathcal{R}_k}^{-1} \mathbf{E}_{\mathcal{R}_k} \mathbf{E}_{\mathcal{R}_{k-1}} \cdots \mathbf{E}_{\mathcal{R}_1} \mathbf{A} = \mathbf{E}_{\mathcal{R}_1}^{-1} \mathbf{E}_{\mathcal{R}_2}^{-1} \cdots \mathbf{E}_{\mathcal{R}_k}^{-1} \mathbf{I} n = \mathbf{E}_{\mathcal{R}_1}^{-1} \mathbf{E}_{\mathcal{R}_2}^{-1} \cdots \mathbf{E}_{\mathcal{R}_k}^{-1}$$

Now note that on the left hand side the product collapses to simply $\mathbf{A}$, and thus

$$\mathbf{A} = \mathbf{E}_{\mathcal{R}_1}^{-1} \mathbf{E}_{\mathcal{R}_2}^{-1} \cdots \mathbf{E}_{\mathcal{R}_k}^{-1}$$

is a product of elementary matrices.

$(e) \quad \Rightarrow \quad (a)$: Assume (e) is true and

$$\mathbf{A} = \mathbf{E}_{\mathcal{R}_1} \cdots \mathbf{E}_{R_k}$$

is a presentation of $\mathbf{A}$ as a product of elementary matrices. Then because each of the matrix factor $\mathbf{E}_{\mathcal{R}_i}$ is invertible, elementary matrix $\left(\mathbf{E}_{\mathcal{R}_i} \right)^{-1}$, $\mathbf{A}$ is also invertible and, moreover,

$$\mathbf{A}^{-1} = \left(\mathbf{E}_{\mathcal{R}_1} \mathbf{E}_{\mathcal{R}_2} \cdots \mathbf{E}_{\mathcal{R}_k} \right)^{-1} = \left(\mathbf{E}_{\mathcal{R}_k} \right)^{-1} \cdots \left(\mathbf{E}_{\mathcal{R}_2} \right)^{-1} \left(\mathbf{E}_{\mathcal{R}_1} \right)^{-1}$$

THEOREM 6.9.

COROLLARY 6.10. *Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Then $\mathbf{AB} = \mathbf{I}$ if and only if $\mathbf{BA} = \mathbf{I}$.*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices and suppose $\mathbf{AB} = \mathbf{I}$. Consider the equation

$$\mathbf{Bx} = \mathbf{0} \quad .$$

Multiplying this equation from the left by $\mathbf{A}$ we get

$$\mathbf{ABx} = \mathbf{A0} = \mathbf{0}$$

while, on the other hand,

$$\mathbf{ABx} = \mathbf{I}_n \mathbf{x} = \mathbf{x}$$

and so if $\mathbf{Bx} = \mathbf{0}$ we must have $\mathbf{x} = 0$. In view of the equivalence of statements (c) and (a) of the Fundamental Theorem, we can conclude that $\mathbf{B}$ is invertible. Hence, $\mathbf{B}^{-1}$ exists and so can multiply the equation $\mathbf{AB} = \mathbf{I}$ from the right by $\mathbf{B}^{-1}$ to get

$$\mathbf{ABB}^{-1} = \mathbf{IB}^{-1} = \mathbf{B}^{-1}$$

and, on the other hand,

$$\mathbf{ABB}^{-1} = \mathbf{AI} = \mathbf{A}$$

and so we must have

$$\mathbf{A} = \mathbf{B}^{-1} \quad .$$

But now
$$\mathbf{BA} = \mathbf{BB}^{-1} = \mathbf{I}$$
which is what we sought to demonstrate.

5. Calculation of Matrix Inverses

Let us now turn to the problem of calculating the inverse of a square matrix. Suppose we start with a matrix $\mathbf{A}$ and apply elementary row operations until we produce a matrix that is not only in reduced row-echelon form, but in fact the identity matrix. This would imply that there would be a corresponding sequence $\{\mathbf{E}_i\}$ of elementary matrices such that
$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}$$
In other words,
$$\mathbf{A}^{-1} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1$$
However,
$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1 = (\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1)\,\mathbf{I}$$
So we can obtain $\mathbf{A}^{-1}$ by applying the same sequence of row operations to the identity matrix $\mathbf{I}$.

This observation leads us to the following procedure.

(1) Form the augmented matrix $[\mathbf{A} \mid \mathbf{I}]$.
(2) Apply the Gauss-Jordan method to attempt to reduce $[\mathbf{A} \mid \mathbf{I}]$ to the form $[\mathbf{I} \mid \mathbf{C}]$.
(3) If successful, then $\mathbf{C} = \mathbf{A}^{-1}$. Otherwise, $\mathbf{A}^{-1}$ does not exist.

EXAMPLE 6.11. Calculate the inverse of
$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

We set
$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]$$
We'll use the notation $R_i \rightarrow R_i + (k)R_j$ to indicate the elementary row operation corresponding to replacing the i^{th} row with its sum with k times the j^{th} row:
$$\begin{aligned} R_2 &\rightarrow R_2 + (-2)R_1 \quad \Rightarrow \quad \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \\ R_1 &\rightarrow R_1 - R_2 \quad \Rightarrow \quad \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right] \end{aligned}$$
The matrix in the first block is now the 2×2 unit matrix. The matrix in the second block should then be the inverse of $\mathbf{A}$. Let's confirm this:
$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} &= \begin{bmatrix} (1,1)\cdot(3,-2) & (1,1)\cdot(-1,1) \\ (2,3)\cdot(3,-2) & (2,3)\cdot(-1,1) \end{bmatrix} \\ &= \begin{bmatrix} 3-2 & -1+1 \\ 6-6 & -2+3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\equiv \mathbf{I} \end{aligned}$$
Hence,
$$\mathbf{A}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$