LECTURE 4

Matrix Algebra

1. Other Matrix Operations

DEFINITION 4.1. *Let* $\mathbf{A}$ *be an* m *by* n *matrix, and let* r *be any real number. Then the* ***scalar product*** $r\mathbf{A}$ *is defined as the* m *by* n *matrix whose* ij^{th} *entry is* r *times the* ij^{th} *entry of* $\mathbf{A}$*:*

$$(r\mathbf{A})_{ij} = r(\mathbf{A})_{ij}$$

EXAMPLE 4.2. If

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

then

$$-2\mathbf{A} = \begin{bmatrix} -2 & 2 \\ -4 & -6 \end{bmatrix}$$

DEFINITION 4.3. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* m *by* n *matrices. Then the* ***matrix sum*** $\mathbf{A}+\mathbf{B}$ *is defined as the* m *by* n *matrix whose* ij^{th} *entry is the sum of the* ij^{th} *entries of* $\mathbf{A}$ *and* $\mathbf{B}$*:*

$$(\mathbf{A}+\mathbf{B})_{ij} = (\mathbf{A})_{ij} + (\mathbf{B})_{ij}$$

EXAMPLE 4.4. If

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

then

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}$$

Combining these two operations of scalar multiplication and addition we can now form **linear combinations** of matrices; e.g. $2\mathbf{A} - 3\mathbf{B}$.

DEFINITION 4.5. *Let* $\mathbf{A}$ *be an* m *by* n *matrix, then the transpose* $\mathbf{A}^T$ *of* $\mathbf{A}$ *is the* n *by* m *such that*

$$\left(\mathbf{A}^T\right)_{ij} = (\mathbf{A})_{ji}$$

In other words, the entries in the i^{th} *row of* $\mathbf{A}^T$ *are identical to the entries in the* i^{th} *column of* $\mathbf{A}$*.*

EXAMPLE 4.6. If

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 1 \\ 3 & -1 \end{bmatrix}$$

then

$$\mathbf{A}^T = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$$

EXAMPLE 4.7. Recall that we can interprete an n-dimenional $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ either as a n by 1 matrix (which we called a column vector)

$$\mathbf{c} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

or as a 1 by n matrix (which we called a row vector)

$$\mathbf{r} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

Note that

$$\mathbf{c} = \mathbf{r}^T$$

and

$$\mathbf{r} = \mathbf{c}^T$$

DEFINITION 4.8. *An n by n matrix with the property that $\mathbf{A} = \mathbf{A}^T$ is called a **symmetric matrix**.*

EXAMPLE 4.9.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

is a symmetric matrix, but

$$\mathbf{B} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

is not symmetric because, for example

$$2 = (\mathbf{B})_{21} \neq \left(\mathbf{B}^T\right)_{21} \equiv (\mathbf{B})_{12} = 1$$

With a little experience it is easy to glance a matrix and determine whether or not it's symmetric.

THEOREM 4.10. *Suppose the matrix product $\mathbf{AB}$ is defined, then*

$$\begin{aligned} (\mathbf{A}^T)^T &= \mathbf{A} \\ (r\mathbf{A})^T &= r\mathbf{A}^T \\ (\mathbf{A}+\mathbf{B})^T &= \mathbf{A}^T + \mathbf{B}^T \\ (\mathbf{AB})^T &= \mathbf{B}^T\mathbf{A}^T \end{aligned}$$