Math 3013
Problem Set 5

1. Determine which of the following mappings are linear transformations.

(a) $T:\mathbb{R}^3\to\mathbb{R}^2 : T([x_1,x_2,x_3]) = [x_1+x_2, x_1-3x_2]$

(b) $T:\mathbb{R}^3\to\mathbb{R}^4 : T([x_1,x_2,x_3]) = [0,0,0,0]$

(c) $T:\mathbb{R}^3\to\mathbb{R}^4 : T([x_1,x_2,x_3]) = [1,1,1,1]$

(d) $T:\mathbb{R}^2\to\mathbb{R}^3 : T([x_1,x_2]) = [x_1-x_2, x_2+1, 3x_1-2x_2]$

2. For each of the following, assume T is a linear transformation, from the data given, compute the specified value.

(a) Given $T([1,0]) = [3,-1]$, and $T([0,1]) = [-2,5]$, find $T([4,-6])$.

(b) Given $T([1,0,0]) = [3,1,2]$, $T([0,1,0]) = [2,-1,4]$, and $T([0,0,1]) = [6,0,1]$, find $T([2,-5,1])$.

3. Find the standard matrix representations of the following linear transformations.

(a) $T:\mathbb{R}^2\to\mathbb{R}^3\ :\ T([x_1,x_2]) = [x_1+x_2, x_1-3x_2, x_1+x_2]$

(b) $T:\mathbb{R}^3\to\mathbb{R}^2\ :\ T([x_1,x_2,x_3]) = [x_1+x_2+x_3, x_1+x_2]$

(c) $T:\mathbb{R}^3\to\mathbb{R}^2\ :\ T([x_1,x_2,x_3]) = [x_1+x_2+x_3, 2x_1+2x_2+2x_3]$

4. For each of the linear transformations $T:\mathbb{R}^m\to\mathbb{R}^n$ in Problem 3, determine

$$\mathit{Range}(T) := \{\mathbf{y}\in\mathbb{R}^n \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x}\in\mathbb{R}^m\}$$

and

$$\mathit{Kernel}(T) := \{\mathbf{x}\in\mathbb{R}^m \mid T(\mathbf{x}) = \mathbf{0}_{\mathbb{R}^n}\} \quad .$$

5. If $T:\mathbb{R}^2\to\mathbb{R}^3$ is defined by $T([x_1,x_2]) = [2x_1+x_2, x_1, x_1-x_2]$ and $T':\mathbb{R}^3\to\mathbb{R}^2$ is defined by $T'([x_1,x_2,x_3]) = [x_1-x_2+x_3, x_1+x_2]$, find the standard matrix representation for the linear transformation $T'\circ T$ that carries $\mathbb{R}^2$ into $\mathbb{R}^2$. Find a formula for $(T'\circ T)([x_1,x_2])$.

6. Determine whether the following statements are *true* or *false*.

(a) Every linear transformation is a function.

(b) Every function mapping $\mathbb{R}^n$ to $\mathbb{R}^m$ is a linear transformation.

(c) Composition of linear transformations corresponds to multiplication of their standard matrix representations.

(d) Function composition is associative.

(e) An invertible linear transformation mapping $\mathbb{R}^n$ to itself has a unique inverse.

(f) The same matrix may be the standard matrix representation for several different linear transformations.

(g) A linear transformation having an $m\times n$ matrix as its standard matrix representation maps $\mathbb{R}^n$ into $\mathbb{R}^m$.

(h) If T and T' are different linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then we may have $T(\mathbf{e}_i) = T'(\mathbf{e}_i)$ for all standard basis vectors $\mathbf{e}_i$ of $\mathbb{R}^n$.

(i) If T and T' are different linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then we may have $T(\mathbf{e}_i) = T'(\mathbf{e}_i)$ for some standard basis vectors $\mathbf{e}_i$ of $\mathbb{R}^n$.

(j) If $B = \{\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$ and T and T' are linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$, then $T(\mathbf{x}) = T'(\mathbf{x})$ for all $\mathbf{x}\in\mathbb{R}^n$ if and only if $T(\mathbf{b}_i) = T'(\mathbf{b}_i)$ for $i = 1,2,\ldots,n$.