Math 3013
SOLUTIONS TO FINAL EXAM
May 16, 2004

1. For each of the following augmented matrices, describe the solution space of the corresponding linear system. (Determine if there are solutions; and, if there are solutions, how many free parameters are needed to describe the general solution.)

(a) (3 pts) $\left[\begin{array}{cccc|c} 3 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ no solution

(b) (3 pts) $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 2 & 1 \\ 0 & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ no solution

(c) (3 pts) $\left[\begin{array}{cccc|c} -3 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right]$ unique solution (a pivot in every column)

(d) (3 pts) $\left[\begin{array}{cccc|c} 2 & 1 & 3 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 3 \end{array}\right]$ one-parameter family of solutions (one column without a pivot)

2. (8 pts) Compute the inverse of

$$\mathbf{A} = \left[\begin{array}{ccc} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{array}\right]$$

- r|

$$\mathbf{A}^{-1} = \left[\begin{array}{ccc} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{array}\right]$$

□

3. (10 pts) Determine if the set $W = \left\{[x, y] \in \mathbb{R}^2 \mid x + y = 1\right\}$ is a subspace of $\mathbb{R}^2$.

- If W is a subspace it must be closed under scalar multiplication: i.e., for every $\lambda \in \mathbb{R}$ and every $\mathbf{v} \in W$, λv must be an element of W. Suppose $\lambda = 0$ and $\mathbf{v} = [x, 1-x] \in W$. Then $\lambda\mathbf{v} = [0, 0]$; but $[0, 0] \notin W$ since $0 + 0 \neq 1$. Hence, W is not closed under scalar multiplication, so it is not a subspace. □

4. Consider the vectors $\{[1,1,0,1],[0,1,1,1],[2,1,-1,1],[-1,0,0,1]\} \in \mathbb{R}^4$

(a) (5 pts) Determine if these vectors are linearly independent.

- The vectors will be linearly independent if and only if the row space of the following matrix is four-dimensional

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Evidently, the row space is three-dimensional, and so the original set of vectors are not linearly independent. $\square$

(b) (5 pts) What is the dimension of the subspace generated by these vectors?

- $$3 \qquad \text{(same as the dimension of the row space of the above matrix)}$$

$\square$

5. Consider the following linear transformation: $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 : T([x_1, x_2) = [x_1 - x_2, x_1 + x_2, x_2]$.

(a) (5 pts) Find a matrix that represents T.

- $$\mathbf{A}_T = \begin{bmatrix} | & | \\ T([1,0]) & T([0,1]) \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\square$

(b) (5 pts) Find a basis for the range of T.

- $Range(T) = ColSp(\mathbf{A}_T)$. It's pretty clear from the form of $\mathbf{A}_T$ that upon reduction to row echelon form, both columns will contain pivots. So the two columns of $\mathbf{A}_T$ will provide a basis for its column space, hence a basis for the range of T.

$$Range(T) = span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right)$$

$\square$

(c) (5 pts) Find a basis for the kernel of T (i.e. the set of vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$).

- $Ker(T) = NullSp(\mathbf{A}_T)$. If we reduce $\mathbf{A}_T$ to reduced row echelon form we obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and so the only solution of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$ will be $\mathbf{x} = \mathbf{0}$. Hence, $Ker(T) = \{\mathbf{0}\}$. $\square$

6. (10 pts) Find a basis for the subspace of $\mathcal{P}_3$ generated by $p_1 = 1 + x + x^2 + x^3$, $p_2 = 1 - x - x^2 - x^3$, $p_3 = 1 - x^3$, and $p_4 = 1 - 2x - 2x^2 - 2x^3$.

- We'll use the standard isomorphism $i : \mathcal{P}_3 \rightarrow \mathbb{R}^4 : \quad a_o + a_1x + a_2x^2 + a_3x^3 \mapsto [a_0, a_1, a_2, a_3]$ to answer this question. Set $\mathbf{v}_1 = i(p_1) = [1,1,1,1]$, $\mathbf{v}_2 = i(p_2) = [1,-1,-1,-1]$, $\mathbf{v}_3 = i(p_3) = [1,0,0,-1]$, and $\mathbf{v}_4 = i(p_4) = [1,-2,-2,-2]$. We'll find a basis for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and then convert that set of vectors back into polynomials. To find a basis for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ we'll arrange them as the rows of a matrix and then find a basis for the row space of that matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & -2 & -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 \\ 0 & -1 & -1 & -2 \\ 0 & -3 & -3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the row space of the matrix is spanned by $\{[1,1,1,1], [0,1,1,1], [0,0,0,1]\}$ Applying i^{-1} to these vectors we arrive at following basis for $span\{p_1, p_2, p_3, p_4\}$

$$\{1 + x + x^2 + x^3, x + x^2 + x^3, x^3\}$$

$\square$

7. Consider the ordered basis $B = \left(x^2, x, 1\right)$ of $\mathcal{P}_2$.
(a) (5 pts) Find the coordinate vector of $3x^2 - x + 2$ relative to B.

- Write $\mathbf{b}_1 = x^2$, $\mathbf{b}_2 = x$, $\mathbf{b}_3 = 1$. Then

$$3x^2 - x +_2 2 = (3)\mathbf{b}_1 + (-1)\,\mathbf{b}_2 + (2)\,\mathbf{b}_3 \quad \Rightarrow \quad \mathbf{v}_B = [3, -1, 2]$$

□

(b) (5 pts) Find the matrix representing the linear transformation $T(p) = x\frac{d^2p}{dx^2} + \frac{dp}{dx} + p$, relative to the basis B.

- We have

$$\begin{aligned} T(\mathbf{b}_1) &= \left[x\frac{d^2}{dx^2} + \frac{d}{dx} + 1\right] x^2 = x\,(2) + 2x + x^2 = (1)\mathbf{b}_1 + (4)\,\mathbf{b}_2 + (0)\,\mathbf{b}_3 \\ T(\mathbf{b}_2) &= \left[x\frac{d^2}{dx^2} + \frac{d}{dx} + 1\right] x = 0 + 1 + x = (0)\,\mathbf{b}_1 + (1)\,\mathbf{b}_2 + (1)\,\mathbf{b}_3 \\ T(\mathbf{b}_3) &= \left[x\frac{d^2}{dx^2} + \frac{d}{dx} + 1\right] 1 = 0 + 0 + 1 = (0)\,\mathbf{b}_1 + (0)\,\mathbf{b}_2 + (1)\,\mathbf{b}_3 \end{aligned}$$

$$\mathbf{A}_T = \begin{bmatrix} | & | & | \\ (T(\mathbf{b}_1))_B & (T(\mathbf{b}_2))_B & (T(\mathbf{b}_3))_B \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

□

8. (10 pts) Let C be the vector space of continuous functions on the real line and consider the function $T : C \to C$ defined by

$$T(f) = x^2\frac{df}{dx} + x$$

Determine if T a linear transformation.

- If T is linear it must satisfy $T(\lambda f) = \lambda T(f)$ for any real number λ and any $f \in C$. Suppose we take $\lambda = 0$. Then

$$\begin{aligned} T(\lambda f) &= T(0 \cdot f) = T(0) = 0 + x = x \\ \lambda T(f) &= 0 \cdot T(f) = 0 \end{aligned}$$

So $T(\lambda f) \neq \lambda T(f)$, hence the transformation T is not linear. □

9. Calculate the following determinants (5 pts each).

(a) –

$$\begin{aligned} det\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} &= (1)(-1)^{1+1}\det\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + (2)(-1)^{1+2}\det\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} + (1)(-1)^{1+3}\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \\ &= +(6-1) - 2(9-2) + (3-1) = 5 - 14 - 1 \\ &= -10 \end{aligned}$$

□

(b) –

$$\begin{aligned} det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix} &= 1(-1)^{1+1}\det\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 0 + 0 + 0 = +2\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) + 0 + 0 \\ &= 2(0-1) \\ &= -2 \end{aligned}$$

□

(c)

$$det\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} = (2)(3)(1)(-1) = -6$$

□

10. (10 pts) Use Crammer's Rule to find a formula for the solution of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$ (i.e. express x_1 and x_2 in terms of a, b, c, d, e, and f).

- We have $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{B}_1 = \begin{bmatrix} e & b \\ f & d \end{bmatrix}$, $\mathbf{B}_2 = \begin{bmatrix} a & e \\ c & f \end{bmatrix}$ and Crammer's Rule gives us

$$\begin{aligned} x_1 &= \frac{\det(\mathbf{B}_1)}{\det(\mathbf{A})} = \frac{ed - bf}{ad - bc} \\ x_2 &= \frac{\det(\mathbf{B}_2)}{\det(\mathbf{A})} = \frac{af - ec}{ad - bc} \end{aligned}$$

□

11. (10 pts) Find the eigenvalues and the eigenvectors of the following matrix: $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

- The characteristic polynomial of $\mathbf{A}$ is $p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & -\lambda \end{bmatrix}\right) = (1-\lambda)^2\lambda$. The eigenvalues of $\mathbf{A}$ are the roots of $p_{\mathbf{A}}(\lambda) = 0$ which are evidently $\lambda = 1$ and $\lambda = 0$.
- To find a basis for the eigenspace corresponding to the eigenvalue $\lambda = 1$ we look for a basis for

$$\begin{aligned} NullSp(\mathbf{A} - (1)\mathbf{I}) &= NullSp\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = NullSp\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \left\{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_2 = 0 \text{ and } x_3 = 0\right\} \\ &= span\{[1, 0, 0]\} \end{aligned}$$

- To find a basis for the eigenspace corresponding to the eigenvalue $\lambda = 0$ we look for a b$NullSp(\mathbf{A} - (1)\mathbf{I}) = NullSp\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = NullSp\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \left\{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_2 = 0 \text{ and } x_3 = 0\right\}$asis for

$$\begin{aligned} NullSp(\mathbf{A} - (0)\mathbf{I}) &= NullSp\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = NullSp\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \left\{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 = x_3 \text{ and } x_2 = -x_3\right\} \\ &= span\{[1, -1, 1]\} \end{aligned}$$

□

12. (10 pts) Let $\mathbf{A}$ be the matrix $\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$. Find a 2×2 matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$.

- First we find the eigenvectors and eigenvalues of $\mathbf{A}$.

$$\begin{aligned} 0 &= \det(\mathbf{A} - \lambda\mathbf{I}) = \det\left(\begin{bmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{bmatrix}\right) = (-1-\lambda)(4-\lambda) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) \\ \Rightarrow & \quad \lambda = 1, 2 \end{aligned}$$

- $\lambda = 1:$

$$\begin{aligned} NullSp(\mathbf{A} - (1)\mathbf{I}) &= NullSp\left(\begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right) = \left\{[x_1, x_2] \in \mathbb{R}^2 \mid x_1 = x_2\right\} \\ &= span\{[1, 1]\} \end{aligned}$$

- $\lambda = 2$

$$\begin{aligned} NullSp(\mathbf{A} - (2)\mathbf{I}) &= NullSp\left(\begin{bmatrix} -3 & 2 \\ -3 & 2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{bmatrix}\right) = \left\{[x_1, x_2] \in \mathbb{R}^2 \mid x_1 = \frac{2}{3}x_2\right\} \\ &= span\left\{\left[\frac{2}{3}, 1\right]\right\} \end{aligned}$$

- $$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad , \quad \mathbf{C} = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} 1 & \frac{2}{3} \\ 1 & 1 \end{bmatrix}$$

□

13. (20 pts) Mark each of the following statements True or False. (Think carefully.)

__T__ (a) If $\mathbb{A}$ is an $n \times m$ matrix, then the linear system $\mathbf{Ax} = \mathbf{0}$ always has at least one solution.

__T__ (b) If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are invertible $n \times n$ matrices, then $\mathbf{AC} = \mathbf{BC}$ implies $\mathbf{A} = \mathbf{B}$.

__T__ (c) If a consistent linear system has fewer equations than unknowns, then there cannot be a unique solution.

__F__ (d) If a consistent linear system has more equations than unknowns, then there will always be a unique solution.

__F__ (e) If a square linear system $\mathbf{Ax} = \mathbf{b}$ has a solution, then $\mathbf{A}$ must have an inverse.

__T__ (f) Every matrix can be transformed, by a sequence of elementary row operations, to a matrix in reduced row echelon form,

__T__ (g) Every basis for a subspace W of $\mathbb{R}^n$ has the same number of vectors.

__F__ (h) If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, then $\det(\mathbf{A} + \mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$.

__T__ (i) If $\mathbf{A}$ is a square matrix in reduced row echelon form, then $\det(\mathbf{A}) = \det\left(\mathbf{A}^3\right)$.

__F__ (j) If $\det(\mathbf{A}) = 0$ then $\mathbf{A}$ is invertible.