Math 3013
Homework Set 7

Problems from §4.1 (pgs. 248-249 of text): 6,7,8,9,12,19

1. (Problems 4.1.6, 4.1.7, 4.1.8, and 4.1.9 in text) Calculate the following determinants.

(a) $\begin{vmatrix} 1 & 4 & -2 \\ 5 & 13 & 0 \\ 2 & -1 & 3 \end{vmatrix}$

- $$\begin{aligned} \begin{vmatrix} 1 & 4 & -2 \\ 5 & 13 & 0 \\ 2 & -1 & 3 \end{vmatrix} &= (1)\begin{vmatrix} 13 & 0 \\ -1 & 3 \end{vmatrix} - (4)\begin{vmatrix} 5 & 0 \\ 2 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 5 & 13 \\ 2 & -1 \end{vmatrix} \\ &= (39-0) - 4(15-0) - 2(-5-26) \\ &= 41 \end{aligned}$$

(b) $\begin{vmatrix} 2 & -5 & 3 \\ 1 & 3 & 4 \\ -2 & 3 & 7 \end{vmatrix}$

- $$\begin{aligned} \begin{vmatrix} 2 & -5 & 3 \\ 1 & 3 & 4 \\ -2 & 3 & 7 \end{vmatrix} &= (2)\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} - (5)\begin{vmatrix} 1 & 4 \\ -2 & 7 \end{vmatrix} + (3)\begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} \\ &= 2(21-12) - 5(7+8) + 3(3+6) \\ &= 120 \end{aligned}$$

(c) $\begin{vmatrix} 1 & -2 & 7 \\ 0 & 1 & 4 \\ 1 & 0 & 3 \end{vmatrix}$

- $$\begin{aligned} \begin{vmatrix} 1 & -2 & 7 \\ 0 & 1 & 4 \\ 1 & 0 & 3 \end{vmatrix} &= (1)\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} + (7)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ &= (3-0) + 2(0-4) + 7(0-1) \\ &= -12 \end{aligned}$$

(d) $\begin{vmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \\ 2 & 1 & -4 \end{vmatrix}$

- $$\begin{aligned} \begin{vmatrix} 2 & -1 & 1 \\ -1 & 0 & 3 \\ 2 & 1 & -4 \end{vmatrix} &= (2)\begin{vmatrix} 0 & 3 \\ 1 & -4 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 3 \\ 2 & -4 \end{vmatrix} + (1)\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} \\ &= 2(0-3) + (4-6) + (-1-0) \\ &= -9 \end{aligned}$$

2. (Problem 4.1.12 in text) Show by direct calculation that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- We calculate explicitly the left-hand-side (L.H.S.) and the right-hand-side (R.H.S.) of this equation.

$$\begin{aligned} L.H.S. &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_2) + a_3 (b_1 c_2 - b_2 c_1) \end{aligned}$$

$$\begin{aligned} R.H.S. &= -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= -a_1 \begin{vmatrix} c_2 & c_3 \\ b_2 & b_3 \end{vmatrix} + a_2 \begin{vmatrix} c_1 & c_3 \\ b_1 & b_3 \end{vmatrix} - a_3 \begin{vmatrix} c_1 & c_2 \\ b_1 & b_2 \end{vmatrix} \\ &= -a_1 (b_3 c_2 - b_2 c_3) + a_2 (b_3 c_1 - b_1 c_3) - a_3 (b_2 c_1 - b_1 c_2) \\ &= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_2) + a_3 (b_1 c_2 - b_2 c_1) \\ &= L.H.S. \end{aligned}$$

3. Mark each of the following *True* or *False*.

(a) The determinant of a 2×2 matrix is a vector.

- False. (Unless you interprete a real number as a 1-dimensional vector.)

(b) If two rows of a 3×3 matrix are interchanged, the sign of the determinant is changed.

- True. (The sign of the determinant flips under a row interchange.)

(c) The determinant of a 3×3 matrix is zero if two rows of the matrix are parallel vectors in $\mathbb{R}^3$.

- True. (In order for the determinant to be non-zero, all the rows have to be linearly independent).

(d) In order for the determinant of a 3×3 matrix to be zero, two rows must be parallel vectors in $\mathbb{R}^3$.

- False. (If row 3 is a linear combination of row 1 and row 2 then $\det(\mathbf{A}) = 0$. See preceding answer.)

(e) The determinant of a 3×3 matrix is zero if the points in $\mathbb{R}^3$ lie in a plane.

- False. (Any three vectors determine a plane, but that does not mean these vectors are linearly independent.)

(f) The determinant of a 3×3 matrix is zero if the points in $\mathbb{R}^3$ lie in a plane through the origin.

- True. (A plane through the origin is a 2-dimensional subspace. If three vectors all live in a 2-dimensional subspace, they must be linearly dependent.)

(g) The parallelogram in $\mathbb{R}^2$ determined by non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ is a square if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

- False. (One can conclude from $\mathbf{a} \cdot \mathbf{b} = 0$ that the sides of the parallelogram are perpendicular; but they need not be of equal length.)

(h) The box in $\mathbb{R}^3$ determined by vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is a cube if and only if $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c} = 0$ and $\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{c}$.

- True. (The first condition implies the sides are all perpedicular, the second that they are all of equal length.)

(i) If the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ is $\pi/4$, then $\|\mathbf{a} \times \mathbf{b}\| = |\mathbf{a} \cdot \mathbf{b}|$.

- True. ($\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \, |\sin(\theta)|$, $|\mathbf{a} \cdot \mathbf{b}| = \|\mathbf{a}\| \, \|\mathbf{b}\| \, |\cos(\theta)|$, and $\cos(\pi/4) = \sin(\pi/4)$.)

(j) For any vector $\mathbf{a}$ in $\mathbb{R}^3$ we have $\|\mathbf{a} \times \mathbf{a}\| = \|\mathbf{a}\|^2$.

- False. (For any vector $\mathbf{a}$ we have $\|\mathbf{a} \times \mathbf{a}\| = 0$.)