Math 3013
Problem Set 6

Problems from §3.1 (pgs. 189-190 of text): 11,16,18

Problems from §3.2 (pgs. 140-141 of text): 4,8,12,23,25,26

1. (Problems 3.1.11 and 3.1. 16 in text). Determine whether the given set is closed under the usual operations of addition and scalar multiplication, and is a (real) vector space.

(a) The set of all diagonal $n \times n$ matrices.

- Let $\mathbf{A} = [a_{ij}]$ be a diagonal $n \times n$ matrix and λ a real number. Then
$$\lambda \mathbf{A} = \lambda \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & 0 & \cdots & 0 \\ 0 & \lambda a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda a_{nn} \end{bmatrix}$$
is also diagonal. So the set of diagonal $n \times n$ matrices is closed under scalar multiplication. Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be two diagonal $n \times n$ matrices. Then
$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}+b_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}+b_{nn} \end{bmatrix}$$
is also diagonal. So the set of diagonal $n \times n$ matrices is also closed under vector addition.

(b) The set P_n of all polynomials in x, with real coefficients and of degree less than or equal to n, together with the zero polynomial.

- Let $p = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $\leq n$ and let λ be a real number. Then
$$\lambda p = \lambda a_n x^n + \lambda a_{n-1} x^{n-1} + \cdots + \lambda a_1 x + \lambda a_0$$
is also a polynomial of degree $\leq n$. Hence, the set P_n is closed under scalar multiplication. Let
$$\begin{aligned} p &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ p' &= = a'_n x^n + a'_{n-1} x^{n-1} + \cdots + a'_1 x + a'_0 \end{aligned}$$
be two polynomials in P_n. Then
$$p + p' = (a_n + a'_n)\, x^n + \left(a_{n-1} + a'_{n-1}\right) x^{n-1} + \cdots + (a_1 + a'_1)\, x + (a_0 + a'_0)$$
is also a polynomial of degree $\leq n$. So the set P_n is closed under vector addition.

2. (Problem 3.1.18 in text). Determine whether the following statements are true or false.

(a) Matrix multiplication is a vector space operation on the set $M_{m\times n}$ of $m \times n$ matrices.

- False. Vector space operations are just scalar multiplication and vector addition.

(b) Matrix multiplication is a vector space operation on the set $M_{n\times n}$ of square $n \times n$ matrices.

- False.

(c) Multiplication of any vector by the zero scalar always yields the zero vector.

- True.

(d) Multiplication of a non-zero vector by a non-zero scalar always yields a non-zero vector.

- True.

(e) No vector is its own additive inverse.

- The zero vector $\mathbf{0}$ is its own additive inverse.

(f) The zero vector is the only vector that is its own additive inverse.

- True.

(g) Multiplication of two scalars is of no concern to the definition of a vector space.

- False. (See Property *S3* on page 181 of the text.)

(h) One of the axioms for a vector space relates the addition of scalars, multiplication of a vector by scalars, and the addition of vectors.

- True. (See Property *S2* on page 181 of the text.)

(i) Every vector spaces has at least two vectors.

- False. The zero vector 0 by itself satisfies all the axioms of a vector space.

(j) Every vector space has at least one vector.

- True. Every vector space contains a zero vector.

3. (Problem 3.2.4 in text). Determine whether the set of all functions f such that $f(1) = 0$ is a subspace of the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$.

- We need to check whether this subset is closed under scalar multiplication and vector addition. Suppose f is a function satisfying $f(1) = 0$ and λ is a real number. Then

  $$(\lambda f)(1) \equiv \lambda f(1) = 0$$

  So this subset is closed under scalar multiplication. Now suppose $f(1) = 0$ and $g(1) = 0$. Then

  $$(f+g)(1) \equiv f(1) + g(1) = 0 + 0 = 0$$

  So this subset is also closed under vector addition. Hence, it is a subspace of the vector space of functions mapping $\mathbb{R}$ into $\mathbb{R}$.

4. (Problem 3.2.8 in text). Let P be the vector space of polynomials. Prove that $span(1, x) = span(1 + 2x, x)$.

- Let

$$p = a_1 x + a_2$$

be an arbitrary polynomial in $span\,(1, x)$. To show that $p \in span(1 + 2x, x)$ we must find coefficients c_1 and c_2 such that

$$p = c_1(1 + 2x) + c_2(x)$$

i.e., we must solve

$$c_1 + 2c_1 x + c_2 x = a_1 x + a_2$$

or

$$\begin{array}{c} 2c_1 + c_2 = a_1 \\ c_1 = a_2 \end{array} \quad \Rightarrow \quad \begin{array}{c} c_1 = a_2 \\ c_2 = \frac{1}{2}(a_1 - a_2) \end{array}$$

Since such a solution always exists, every $p \in span\,(1, x)$ lies also in $span\,(1 + 2x, x)$. So

$$span\,(1, x) \subset span\,(1 + 2x, x)$$

It's even easier to show that every $p \in span\,(1 + 2x, x)$ lies also in $span\,(1, x)$;

$$p \in span\,(1 + 2x, x) \quad \Rightarrow \quad p = c_1(1 + 2x) + c_2 x = c_1 + (c_2 + 2c_1)\,x \in span\,(1, x)\,.$$

Hence,

$$span(1 + 2x, x) \subset span\,(1, x)$$

Finally,

$$span\,(1, x) \subset span\,(1 + 2x, x) \quad \text{and} \quad span(1 + 2x, x) \subset span\,(1, x) \qquad \Rightarrow \qquad span(1 + 2x, x) = span\,(1, x)$$

5. (Problem 3.2.12 in text). Determine whether the following set of vectors is dependent or independent:$\{1, 4x + 3, 3x - 4,$ in P.

- Let

$$\begin{aligned} p_1 &= 1 \\ p_2 &= 4x + 3 \\ p_3 &= 3x - 4 \\ p_4 &= x^2 + 2 \\ p_5 &= x - x^2 \end{aligned}$$

If the polynomials are dependent, then (by definition) there must be non-trivial solutions of

(1) $$c_1 p_1 + c_2 p_2 + c_3 p_3 + c_4 p_4 + c_5 p_5 = 0$$

or

$$\begin{aligned} 0 &= c_1 + 4c_2 x + 3c_2 + 3c_3 x - 4c_3 + c_4 x^2 + 2c_4 + c_5 x - c_5 x^2 \\ &= (c_1 + 3c_2 - 4c_3 + 2c_4) + (4c_2 + 3c_3 + c_5)\,x + (c_4 - c_5)\,x^2 \end{aligned}$$

or

$$\begin{aligned} c_1 + 3c_2 - 4c_3 + 2c_4 &= 0 \\ 4c_2 + 3c_3 + c_5 &= 0 \\ c_4 - c_5 &= 0 \end{aligned}$$

This is a system of 3 homogeneous equations in 5 unknowns. Such a system will have *at least* a 2-parameter family of solutions. So we will have non-trivial solutions of (1), hence the polynomials are dependent.

6.. (Problem 3.1.25 in text). Determine whether the following statements are true or false.

(a) The set consisting of the zero vector is a subspace for every vector space.

- True.

(b) Every vector space has at least two distinct subspaces.

- False. The vector space consisting of just the zero vector has no other subspaces.

(c) Every vector space with a nonzero vector has at least two distinct subspaces.

- True. The entire vector space and the (span of the) zero vector are subspaces (which are distinct because there exists a non-zero vector).

(d) If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a subset of a vector space then $\mathbf{v}_i$ is in $span\,(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ for $i = 1, 2, \dots, n$.

- True.

(e) If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a subset of a vector space then $\mathbf{v}_i + \mathbf{v}_j$ is in $span\,(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ for all choices of i and j between 1 and n.

- False. Subsets are not in general closed under vector addition.

(f) If $\mathbf{u} + \mathbf{v}$ lies in a subspace W of a vector space V, then both **u and v** lie in W.

- False. Consider $W = span([1,1]) \subset \mathbb{R}^2$. Then $[1,0] + [0,1] \in W$ but $[1,0] \notin W$ and $[0,1] \notin W$.

(g) Two subspaces of a vector space may have empty intersection.

- False. Every subspace contains the zero vector; hence, the intersection of two subspaces will always contain at least the zero vector.

(h) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is independent, each vector in V can be expressed uniquely as a linear combination of vectors in S.

- True.

(i) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is independent and generates V, then each vector in V can be expressed uniquely as a linear combination of vectors in S.

- True.

(j) If each vector in V can be expressed uniquely as a linear combination of vectors in $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, then S is an independent set.

- True.

7. (Problem 3.1.26 in text). Let V be a vector space. Determine whether the following statements are true or false.

(a) Every independent set of vectors in V is a basis for subspace the vectors span.

- False. You need sufficiently many independent vectors to have a basis. (E.g., two vectors in $\mathbb{R}^3$ might be linearly independent, but you need three independent vectors to form a basis for $\mathbb{R}^3$.)

(b) If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ generates V, then each $\mathbf{v} \in V$ is a linear combination of vectors in this set.

- True.

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ generates V, then each $\mathbf{v} \in V$ is a unique linear combination of vectors in this set.

- False. The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ need not be linearly independent; so there may be more than one way of writing the zero vector as a linear combination of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$.

(d) If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ generates V and is independent, then each $\mathbf{v} \in V$ is a linear combination of vectors in this set.

- True.

(e) If If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ generates V, then this set of vectors is independent.

- False. (See Part c).

(f) If each vector in V is a unique linear combination of the vectors in the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then this set is independent.

- True.

(g) If each vector in V is a unique linear combination of the vectors in the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then this set is a basis for V.

- True.

(h) All vector spaces having a basis are finitely generated.

- False.

(i) Every independent subset of a finitely generated vector space is a part of some basis for V.

- True.

(j) Any two bases in a finite-dimensional vector space V have the same number of elements.

- True.