Math 3013
Problem Set 5

Problems from §2.1 (pgs. 134-136 of text): 1,3,11,12,13,16,23

Problems from §2.2 (pgs. 140-141 of text): 1,3,5,7,11

Problems from §2.3 (pgs. 152-154 of text): 1,2,3,4,5,7,13,15,19,29

1. (Problem 2.1.1 in text). Give a geometric criterion for a set of two distinct nonzero vectors in $\mathbb{R}^2$ to be dependent.

- If two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent, then there must exist a solution of

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$$

with at least one of the coefficients c_1, c_2 not zero. Suppose (without loss of generality) that $c_2 \neq 0$. Then c_1 can not equal zero either (otherwise we'd have $c_2\mathbf{v}_2 = \mathbf{0}$ with neither c_2 or $\mathbf{v}_2$ zero). Then we can multiply both sides of this equation by $1/c_2$ to obtain

$$\frac{c_1}{c_2}\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0} \quad \Rightarrow \quad \mathbf{v}_2 = -\frac{c_1}{c_2}\mathbf{v}_1$$

So $\mathbf{v}_2$ must be a non-zero scalar multiple of $\mathbf{v}_1$. But then, this implies that $\mathbf{v}_2$ is either parallel (or anti-parallel) to $\mathbf{v}_1$. $\square$

2. (Problem 2.1.3 in text). Give a geometric criterion for a set of two distinct nonzero vectors in $\mathbb{R}^3$ to be dependent.

- By exactly the same reasoning we used in Problem 1, we can conclude that if two distinct non-zero vectors in $\mathbb{R}^3$ are dependent then they must be parallel (or anti-parallel). $\square$

3. (Problem 2.1.11 in text). Find a basis for the subspace spanned by the vectors $[1,2,1,1], [2,1,0,-1], [-1,4,3,8], [0,3,2,$
$\mathbb{R}^4$.

- First we form a 4×4 matrix $\mathbf{A}$ whose columns correspond to the above set of vectors.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 4 & 3 \\ 1 & 0 & 3 & 2 \\ 1 & -1 & 8 & 5 \end{bmatrix}$$

Now we row-reduce $\mathbf{A}$ to row-echelon form.

$$\xrightarrow{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -3 & 6 & 3 \\ 0 & -2 & 4 & 2 \\ 0 & -3 & 9 & 5 \end{bmatrix} \xrightarrow{\substack{R_2 \to -\frac{1}{3}R_2 \\ R_3 \to R_3 - \frac{2}{3}R_2 \\ R_4 \to R_4 - R_2}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivots of the final matrix (a row-echelon form of $\mathbf{A}$) are in the first three columns. Hence, the first three columns

$$\{[1,2,1,1], [2,1,0,-1], [-1,4,3,8]\}$$

of $\mathbf{A}$ will form a basis for the column space

$$ColSp(\mathbf{A}) = span\left([1,2,1,1],[2,1,0,-1],[-1,4,3,8],[0,3,2,5]\right)$$

$\square$

4. (Problem 2.1.12 in text). Find a basis for the column space of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 1 & 7 & 2 \\ 6 & -2 & 0 \end{bmatrix}$$

- We'll apply the same technique used in Problem 3.

$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 1 \\ 1 & 7 & 2 \\ 6 & -2 & 0 \end{bmatrix} \longrightarrow \begin{matrix} 1 & 7 & 2 \\ 5 & 2 & 1 \\ 2 & 3 & 1 \\ 6 & -2 & 0 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 7 & 2 \\ 0 & -33 & -9 \\ 0 & -11 & -3 \\ 0 & -44 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The pivots in the row-echelon form of $\mathbf{A}$ are in the first two columns. Therefore, the first two columns of $\mathbf{A}$

$$\{[2,5,1,6]\,,[3,2,7,-2]\}$$

will form a basis for the column space of $\mathbf{A}$. $\square$

5. (Problem 2.1.13 in text). Find a basis for the row space of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 0 & 4 & 2 \\ 3 & 2 & 8 & 7 \end{bmatrix}$$

The row space of $\mathbf{A}$ is the span of the row vectors $\{[1,3,5,7]\,,[2.0,4,2]\,,[3,2,8,7]\}$ of $\mathbf{A}$ To find a basis for the span of these vectors we arrange them as the columns of a new matrix $\mathbf{A}'$

$$\mathbf{A}' = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \\ 5 & 4 & 8 \\ 7 & 2 & 7 \end{bmatrix}$$

which happens to be the transpose of our original matrix $\mathbf{A}$. We now row-reduce $\mathbf{A}'$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \\ 5 & 4 & 8 \\ 7 & 2 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -7 \\ 0 & -6 & -7 \\ 0 & -12 & -14 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{H}$$

The pivots of $\mathbf{H}$ are contained in the first two columns, therefore the first two columns of $\mathbf{A}'$ form a basis for the column space of $\mathbf{A}'$, which is indentical to row space of our original matrix $\mathbf{A}$. Thus,

$$\{[1,3,5,7]\,,[2,0,4,2]\}$$

is a basis for the row space of $\mathbf{A}$.

6. (Problems 2.1.16 and 2.1.23 in text). Determine whether the following sets of vectors are dependent or independent.

(a) $\{[1,3]\,,[-2,-6]\}$ in $\mathbb{R}^2$.

- Let $\mathbf{v}_1 = [1,3]$ and $\mathbf{v}_2 = [-2,-6]$ and Note that

$$2\mathbf{v}_1 + \mathbf{v}_2 = 2[1,3] + [-2,-6] = [0,0]$$

so $c_1 = 2$ and $c_2 = 1$ is a non-trivial solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Thus, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent. $\square$

(b) $\{[1,-4,3], [3,-11,2], [1,-3,-4]\}$ in $\mathbb{R}^3$.

- Let $\mathbf{v}_1 = [1,-4,3]$, $\mathbf{v}_2 = [3,-11,2]$ and $\mathbf{v}_3 = [1,-3,-4]$. We shall look for non-trivial solutions of

$$\begin{aligned}\mathbf{0} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \\ &= c_1[1,-4,3] + c_2[3,-11,2] + c_3[1,-3,-4] \\ &= [c_1 + 3c_2 + c_3, -4c_1 - 11c_2 - 3c_3, 3c_1 + 2c_2 - 4c_3]\end{aligned}$$

or

(1)
$$\begin{aligned} c_1 + 3c_2 + c_3 &= 0 \\ -4c_1 - 11c_2 - 3c_3 &= 0 \\ 3c_1 + 2c_2 - 4c_3 &= 0\end{aligned}$$

We thus examine the following augmented matrix

$$\mathbf{A} = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -4 & -11 & -3 & 0 \\ 3 & 2 & -4 & 0 \end{array}\right]$$

Row reducing this matrix yields

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -7 & -7 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the original system of equations is equivalent to a system of 2 independent equations in 3 unknowns. This means there will be infinitely many (in fact, a one-parameter family of) solutions of (1). Hence, there are non-trivial solutions so the original set of vectors are linearly independent. $\square$

7. (Problems 2.2.1, 2.2.3, and 2.2.5 in text). For each of the following matrices find the rank of the matrix, a basis for its row space, a basis for its column space, and a basis for its null space.

(a)

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -3 & 1 \\ 3 & 4 & 2 & 2 \end{bmatrix}$$

Let us first row reduce the given matrix to row-echelon form.

$$\rightarrow \begin{bmatrix} 1 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 4 & \frac{13}{2} & -\frac{1}{2} \end{bmatrix} \equiv \mathbf{H}$$

The pivots of this matrix lie in columns 1 and 2. Therefore, first two columns of the orginal matrix $\mathbf{A}$ will form a basis for the column space of $\mathbf{A}$.

$$ColSp(\mathbf{A}) = span([2,3], [0,4])$$

The non-zero rows of the row reduced form $\mathbf{H}$ of $\mathbf{A}$ will be basis for the row space of $\mathbf{A}$. Hence,

$$RowSp(\mathbf{A}) = span\left(\left[1, 0, \frac{-3}{2}, \frac{1}{2}\right], \left[0, 4, \frac{13}{2}, -\frac{1}{2}\right]\right)$$

The null space of $\mathbf{A}$ is the solution set of $\mathbf{Ax}=\mathbf{0}$, which coincides with the solution set of $\mathbf{Hx}=\mathbf{0}$, i.e. the solution set of

$$\begin{array}{lcl} x_1 - \frac{3}{2}x_3 + \frac{1}{2}x_4 = 0 \\ 4x_2 - \frac{13}{2}x_3 - \frac{1}{2}x_4 = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} x_1 = \frac{3}{2}x_3 - \frac{1}{2}x_4 \\ x_2 = \frac{13}{8}x_3 + \frac{1}{8}x_4 \end{array}$$

$$\Rightarrow \quad \mathbf{x} = \begin{bmatrix} \frac{3}{2}x_3 - \frac{1}{2}x_4 \\ \frac{13}{8}x_3 + \frac{1}{8}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} \frac{3}{2} \\ \frac{13}{8} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{8} \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{x} \in span\left(\begin{bmatrix} \frac{3}{2} \\ \frac{13}{8} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{8} \\ 0 \\ 1 \end{bmatrix} \right)$$

so the vectors $\left\{ \left[\frac{3}{2}, \frac{13}{8}, 1, 0\right], \left[-\frac{1}{2}, \frac{1}{8}, 0, 1\right] \right\}$ form a basis for the null space of $\mathbf{A}$. Finally,

$$rank\left(\mathbf{A}\right) = \dim\left(ColSp\left(\mathbf{A}\right)\right) = \dim\left(RowSp\left(\mathbf{A}\right)\right) = 2$$

(b)

$$\mathbf{A} = \begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}$$

- We proceed as in Part (a). The matrix $\mathbf{A}$ is row equivalent to the following matrix in row-echelon form

$$\mathbf{H} = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 6 & 6 & 3 \\ 0 & 0 & 4 & \frac{11}{2} \\ 0 & 0 & 0 & \frac{-19}{9} \end{bmatrix}$$

Each column of $\mathbf{H}$ contains a pivot; so each column of $\mathbf{A}$ is a basis vector for the column space of $\mathbf{A}$. The matrix $\mathbf{H}$ contains no zero rows; so every row of $\mathbf{H}$ is a basis vector for the row space of $\mathbf{A}$. Because the linear system $\mathbf{Ax}=\mathbf{0}$ is equivalent to $\mathbf{Hx}=\mathbf{0}$, and because the latter is evidently a system of 4 independent equations and 4 unknowns; we shall a unique solution to $\mathbf{Ax}=\mathbf{0}$; namely, $\mathbf{x}=\mathbf{0}$. Hence, the null space of $\mathbf{A}$ is spanned by $\mathbf{x}=0$ (i.e., it's zero-dimensional). The rank of $\mathbf{A}$ is equal to the dimension of its column space, which equals the number of vectors in a basis for the column space, which equals 4. □

(c)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

- This matrix row-reduces to

$$\mathbf{H} = \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -1 \end{bmatrix}$$

The pivots of $\mathbf{H}$ lie in the first three columns. Therefore, the first three columns of $\mathbf{A}$ will be a basis for the column space of $\mathbf{A}$. Each row of $\mathbf{H}$ is non-zero; therefore, each row of $\mathbf{H}$ will be a basis vector for the row space of $\mathbf{A}$. The solution set of $\mathbf{Ax} = \mathbf{0}$ coincides with the solution set of $\mathbf{Hx} = \mathbf{0}$, or

$$\begin{array}{l} 2x_1 + x_2 + 2x_4 = 0 \\ x_2 + 2x_3 + x_4 = 0 \\ -3x_3 - x_4 = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} x_1 = -\frac{5}{6}x_4 \\ x_2 = -\frac{1}{3}x_4 \\ x_3 = -\frac{1}{3}x_4 \end{array}$$

So every solution is a vector of the form

$$\mathbf{x} = \begin{bmatrix} -\frac{5}{6}x_4 \\ -\frac{1}{3}x_4 \\ -\frac{1}{3}x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -\frac{5}{6} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \in \mathit{span}\left(\begin{bmatrix} -\frac{5}{6} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right)$$

Thus, $\left\{ \left[-\frac{5}{6}, -\frac{1}{3}, -\frac{1}{3}, 1 \right] \right\}$ is a basis for the null space of $\mathbf{A}$. The rank of $\mathbf{A}$ is equal to the number of basis vectors for the column space of $\mathbf{A}$, which is 3. $\square$

8. (Problem 2.2.7 in text). Determine whether the following matrix is invertible, by finding its rank.

$$\mathbf{A} = \begin{bmatrix} 0 & -9 & -9 & 2 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}$$

- This matrix is row equivalent to

$$\mathbf{H} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -9 & -9 & 2 \\ 0 & 0 & 0 & -\frac{7}{9} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of $\mathbf{A}$ is equal to the dimension of its row space, which is equal to the number of non-zero rows in a row-echelon form of $\mathbf{A}$. Hence, $rank(\mathbf{A}) = 3$. This implies that the dimension of the null space of $\mathbf{A}$ is 1 since

$$\#\text{ of columns of } \mathbf{A} = \text{rank}\,(\mathbf{A}) - \dim\,(\text{null space of } \mathbf{A})$$

Hence, the matrix is not invertible, since it rank does does equal its number of columns (See Theorem 2.6 in text.) $\square$

9. (Problem 2.2.11 in text). Determine whether the following statements are *true* or *false*.

(a) The number of independent row vectors in a matrix is the same as the number of independent column vectors.

- True. This number is the rank of a matrix. $\square$

(b) If $\mathbf{H}$ is a row-echelon form of a matrix $\mathbf{A}$, then the nonzero column vectors of $\mathbf{H}$ form a basis for the column space of $\mathbf{A}$.

- False. It is the columns of $\mathbf{A}$ that corresponding to column vectors of $\mathbf{H}$ that contain pivots that form a basis for the column space of $\mathbf{A}$. $\square$

(c) If $\mathbf{H}$ is a row-echelon form of a matrix $\mathbf{A}$, then the nonzero row vectors of $\mathbf{H}$ from a basis for the row space of $\mathbf{A}$.

- True. $\square$

(d) If an $n \times n$ matrix $\mathbf{A}$ is invertible then $rank\,(\mathbf{A}) = n$.

- True. (See Theorem 2.6 in text.) $\square$

(e) For every matrix $\mathbf{A}$ we have $rank(\mathbf{A}) > 0$.

- False. The rank of a zero matrix will be zero. $\square$

(f) For all positive integers m and n, the rank of an $m \times n$ matrix might be any number from 0 to the maximum of m and n.

- False. We have
$$n = \#\text{ of columns} = rank + \dim(\text{null space})$$
or
$$rank = n - \dim(\text{null space}) \leq n$$
$\square$

(g) For all positive integers m and n, the rank of an $m \times n$ matrix might be any number from 0 to the minimum of m and n.

- True. As in Part (f) we have
$$rank = n - \dim(\text{null space}) \leq n$$
Now if $n > m$, then the equation $\mathbf{Ax} = \mathbf{0}$, would correspond to a consistent linear system with $(n-m)$ more variables (n) then equations (m). Hence, the dimension of the null space would be at least $n-m$. In fact, the dimension of the null space could be as large as n (which would happen, e.g. if the matrix contained all zeros).
  Hence, if $m < n$,
$$n - m \leq \dim(\text{null space}) \leq n$$
Hence
$$0 = n - n \leq rank \leq n - (m - n) = m = \min(m, n)$$
On the other hand, if $m \geq n$, then we dimension of the null space can be as small as 0 (because we will have nore equations than unknowns). The maximum dimension of the null space will still be n (again corresponding to the dimension of the null space of the zero matrix). Thus, if $m \geq n$
$$0 \leq \dim(\text{null space}) \leq n = n$$
Hence,
$$0 = n - n \leq rank \leq n - 0 = n = \min(m, n)$$
We conclude
$$0 \leq rank \leq \min(m, n)$$
$\square$

(h) For all positive integers m and n, the nullity of an $m \times n$ matrix might be any number from 0 to n.

- False. In Part (g) we showed
$$0 \leq rank \leq \min(m, n)$$
But
$$\begin{aligned} \dim(\text{null space}) &= \#\text{ of columns } - rank \\ &= n - rank \end{aligned}$$

So

$$n - \min(m, n) \leq \dim(\text{null space}) \leq n - 0 = n$$

Since $n - \min(m, n) \neq 0$ if $m < n$, the statement is false. $\square$

(i) For all positive integers m and n, the nullity of an $m \times n$ matrix might be any number from 0 to m.

- False. As we argued in Part (h),

$$n - \min(m, n) \leq \dim(\text{null space}) \leq= n$$

Since, $n - \min(m, n) \neq 0$ if $m < n$, the statement is false (the upper boudn is also wrong). $\square$

(j) For all positive integers m and n, with $m \geq n$, the nullity of an $m \times n$ matrix might be any number from 0 to n.

- True. In Part (h) we found

$$n - \min(m, n) \leq \dim(\text{null space}) \leq n - 0 = n$$

But if $m \geq n$, then $n - \min(m, n) = 0$, and so we have

$$0 \leq \dim(\text{null space}) \leq n$$

$\square$

10. (Problems 2.3.1, 2.3.2, 2.3.3, and 2.3.4 in text). Determine which of the following mappings are linear transformations.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_1 + x_2, x_1 - 3x_2]$

- This mapping is linear since if $\mathbf{v} = [x_1, x_2, x_3]$

$$\begin{aligned} T(\lambda \mathbf{v}) &= T(\lambda [x_1, x_2, x_3]) \\ &= T([\lambda x_1, \lambda x_2, \lambda x_3]) \\ &= [\lambda x_1 + \lambda x_2, \lambda x_1 - 3\lambda x_2] \\ &= \lambda [x_1 + x_2, x_1 - 3x_2] \\ &= \lambda T([x_1, x_2, x_3]) \\ &= \lambda T(\mathbf{v}) \quad (T \text{ preserves scalar multiplication}) \end{aligned}$$

and if $\mathbf{v} = [x_1, x_2, x_3]$ and $\mathbf{v}' = [x_1', x_2', x_3']$

$$\begin{aligned} T(\mathbf{v} + \mathbf{v}') &= T([x_1 + x_1', x_2 + x_2', x_3 + x_3']) \\ &= [x_1 + x_1' + x_2 + x_2', x_1 + x_1' - 3(x_2 - x_2')] \\ &= [x_1 + x_2, x_1 - 3x_2] + [x_1' + x_2', x_1' - 3x_2'] \\ &= T(\mathbf{v}) + T(\mathbf{v}') \quad (T \text{ preserves vector addition}) \end{aligned}$$

$\square$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^4 : T([x_1, x_2, x_3]) = [0, 0, 0, 0]$

- This mapping is linear since if $\mathbf{v} = [x_1, x_2, x_3]$

$$\begin{aligned} T(\lambda \mathbf{v}) &= T([\lambda x_1, \lambda x_2, \lambda x_3]) \\ &= [0,0,0,0] \\ &= \lambda [0,0,0,0] \\ &= \lambda T([x_1, x_2, x_3]) \\ &= \lambda T(\mathbf{v}) \quad (T \text{ preserves scalar multiplication}) \end{aligned}$$

and if $\mathbf{v} = [x_1, x_2, x_3]$ and $\mathbf{v}' = [x_1', x_2', x_3']$

$$\begin{aligned} T(\mathbf{v} + \mathbf{v}') &= T([x_1 + x_1', x_2 + x_2', x_3 + x_3']) \\ &= [0,0,0,0] \\ &= [0,0,0,0] + [0,0,0,0] \\ &= T(\mathbf{v}) + T(\mathbf{v}') \quad (T \text{ preserves vector addition}) \end{aligned}$$

□

(c) $T : \mathbb{R}^3 \to \mathbb{R}^4 : T([x_1, x_2, x_3]) = [1,1,1,1]$

- This mapping is not linear since if $\mathbf{v} = [x_1, x_2, x_3]$

$$\begin{aligned} T(\mathbf{v}) &= [1,1,1,1] \\ T(2\mathbf{v}) &= [1,1,1,1] \neq 2[1,1,1,1] = 2T(\mathbf{v}) \end{aligned}$$

So the mapping does not preserve scalar multiplication. □

(d) $T : \mathbb{R}^2 \to \mathbb{R}^3 : T([x_1, x_2]) = [x_1 - x_2, x_2 + 1, 3x_1 - 2x_2]$

- This mapping is not linear since, e.g., if $\mathbf{v} = [1,1,1]$

$$\begin{aligned} T(\mathbf{v}) &= [0,2,1] \\ T(2\mathbf{v}) &= T([2,2,2]) = [0,3,2] \neq [0,4,2] = 2T(\mathbf{v}) \end{aligned}$$

So the mapping does not preserve scalar multiplication. □

11. (Problems 2.3.5 and 2.3.7 in text). For each of the following, assume T is a linear transformation, from the data given, compute the specified value.

(a) Given $T([1,0]) = [3,-1]$, and $T([0,1]) = [-2,5]$, find $T([4,-6])$.

- Because linear transformations preserve scalar multiplication and vector addition, they also preserve linear combinations:

$$T(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2)$$

Now take $\mathbf{e}_1 = [1,0]$ and $\mathbf{e}_2 = [0,1]$. Then

$$\begin{aligned} T([4,-6]) &= T(4\mathbf{e}_1 - 6\mathbf{e}_2) \\ &= 4T(\mathbf{e}_1) - 6T(\mathbf{e}_2) \\ &= 4[3,-1] - 6[-2,5] \\ &= [12 + 12, -4 - 30] \\ &= [24, -34] \end{aligned}$$

□

(b) Given $T([1,0,0]) = [3,1,2]$, $T([0,1,0]) = [2,-1,4]$, and $T([0,0,1]) = [6,0,1]$, find $T([2,-5,1])$.

- As in Part (a), we set $\mathbf{e}_1 = [1,0,0]$, $\mathbf{e}_2 = [0,1,0]$, and $\mathbf{e}_3 = [0,0,1]$ and then compute

$$\begin{aligned} T([2,-5,1]) &= T(2\mathbf{e}_1 - 5\mathbf{e}_2 + \mathbf{e}_3) \\ &= 2T(\mathbf{e}_1) - 5T(\mathbf{e}_2) + T(\mathbf{e}_3) \\ &= 2[3,1,2] - 5[2,-1,4] + [6,0,1] \\ &= [6-10+6, 2+5+0, 4-20+1] \\ &= [2,7,-15] \end{aligned}$$

$\square$

12. (Problems 2.3.13 and 2.3.15 in text). Find the standard matrix representations of the following linear transformations.

(a) $T([x_1,x_2]) = [x_1+x_2, x_1-3x_2]$

- The standard matrix representations are computed by computing the action of the linear transformation T on the standard basis vectors, and then using results as the columns of the corresponding matrix. For the case at hand we have

$$\begin{aligned} \mathbf{e}_1 &= [1,0] \quad \Rightarrow \quad T(\mathbf{e}_1) = [1+0, 1-3(0)] = [1,1] \\ \mathbf{e}_2 &= [0,1] \quad \Rightarrow \quad T(\mathbf{e}_2) = [0+1, 0-3(1)] = [1,-3] \end{aligned}$$

So the matrix corresponding to T is

$$\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$\square$

(b) $T([x_1,x_2,x_3]) = [x_1+x_2+x_3, x_1+x_2, x_1]$

- We proceed as in Part (a).

$$\begin{aligned} \mathbf{e}_1 &= [1,0,0] \quad \Rightarrow \quad T(\mathbf{e}_1) = [1+0+0, 1+0, 1] = [1,1,1] \\ \mathbf{e}_2 &= [0,1,0] \quad \Rightarrow \quad T(\mathbf{e}_2) = [0+1+0, 0+1, 0] = [1,1,0] \\ \mathbf{e}_3 &= [0,0,1] \quad \Rightarrow \quad T(\mathbf{e}_3) = [0+0+1, 0+0, 0] = [1,0,0] \end{aligned}$$

So the matrix corresponding to T is

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

$\square$

13. (Problem 2.3.19 in text). If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $T([x_1,x_2]) = [2x_1+x_2, x_1, x_1-x_2]$ and $T' : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T'([x_1,x_2,x_3]) = [x_1-x_2+x_3, x_1+x_2]$, find the standard matrix representation for the linear transformation $T' \circ T$ that carries $\mathbb{R}^2$ into $\mathbb{R}^2$. Find a formula for $(T' \circ T)([x_1,x_2])$.

- The matrix representations corresponding to T and T' are

$$\mathbf{M}_T = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \quad , \quad \mathbf{M}_{T'} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The matrix representation corresponding to $T' \circ T$ will be given by the product of the corresponding matrices

$$\mathbf{M}_{T' \circ T} = \mathbf{M}_{T'}\mathbf{M}_T = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

Hence

$$(T' \circ T)(x_1, x_2) = [2x_1, 3x_1 + x_2]$$

: □

14. (Problem 2.3.29 in text). Determine whether the following statements are *true* or *false*.

(a) Every linear transformation is a function.

- True. □

(b) Every function mapping $\mathbb{R}^n$ to $\mathbb{R}^m$ is a linear transformation.

- False. In order to be a linear transformation a function $f : \mathbb{R}^n \to \mathbb{R}^m$ must preserve scalar multiplication and vector addtion. □

(c) Composition of linear transformations corresponds to multiplication of their standard matrix representations.

- True. □

(d) Function composition is associative.

- True. □

(e) An invertible linear transformation mapping $\mathbb{R}^n$ to itself has a unique inverse.

- True. (This follows from the corresponding theorem about invertible matrices.) □

(f) The same matrix may be the standard matrix representation for several different linear transformations.

- False. (Unless one allows more general vector spaces - but idea won't be broached until Chapter 3.) □

(g) A linear transformation having an $m \times n$ matrix as its standard matrix representation maps $\mathbb{R}^n$ into $\mathbb{R}^m$.

- True. □

(h) If T and T' are different linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then we may have $T(\mathbf{e}_i) = T'(\mathbf{e}_i)$ for all standard basis vectors $\mathbf{e}_i$ of $\mathbb{R}^n$.

- False. Linear transformations are determined uniquely by their standard matrix representations. □

(i) If T and T' are different linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then we may have $T(\mathbf{e}_i) = T'(\mathbf{e}_i)$ for some standard basis vectors $\mathbf{e}_i$ of $\mathbb{R}^n$.

- True. (So long as they are not all the same.) $\square$

(j) If $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$ and T and T' are linear transformations from $\mathbb{R}^n$ into $\mathbb{R}^m$, then $T(\mathbf{x}) = T'(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ if and only if $T(\mathbf{b}_i) = T'(\mathbf{b}_i)$ for $i = 1, 2, \ldots, n$.

- True. $\square$