Math 3013
Problem Set 4

Problems from §1.6 (pgs. 99-101 of text): 1,3,5,7,9,11,17,19,21,35,37,38

1. (Problems 1,3,4,7,9 in text). Determine whether the indicated subset is a subspace of the given $\mathbb{R}^n$.

(a) $W = \{[r, -r] \mid r \in \mathbb{R}\}$ in $\mathbb{R}^2$

- It suffices to show that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are in W then so is any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Set

$$\mathbf{v}_1 = [r_1, -r_1] \quad , \quad \mathbf{v}_2 = [r_2, -r_2]$$

Then

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= c_1[r_1, -r_1] + c_2[r_2, -r_2] \\ &= [c_1r_1 + c_2r_2, -c_1r_1 - c_2r_2] \\ &= [(c_1r_1 + c_2r_2), -(c_1r_1 + c_2r_2)] \in W \end{aligned}$$

□

(b) $W = \{[n, m] \mid \ n$ and n are integers$\}$ in $\mathbb{R}^2$

- This subset is not closed under scalar multiplication for

$$[1,1] \in W \quad \text{but} \quad \sqrt{2}[1,1] = \left[\sqrt{2}, \sqrt{2}\right] \notin W$$

Since this subset is not closed under scalar multiplication it cannot be a subspace. □

(c) $W = \{[x, y, z] \mid x, y, z \in \mathbb{R}$ and $z = 3x + 2\}$ in $\mathbb{R}^3$

- Consider two arbitrary vectors in W

$$\mathbf{v}_1 = [x_1, y_1, 3x_1 + 2] \quad , \quad \mathbf{v}_2 = [x_2, y_2, 3x_2 + 2]$$

we have

$$\mathbf{v}_1 - \mathbf{v}_2 = [x_1 - x_2, y_1 - y_2, 3(x_1 - x_2) + 0] \notin W$$

Since the difference of two vectors in W does not lie in W, W is not a subspace. □

(d) $W = \{[x, y, z] \mid x, y, z \in \mathbb{R}$ and $z = 1$, $y = 2x\}$ in $\mathbb{R}^3$

- Consider two arbitrary vectors in W

$$\mathbf{v}_1 = [x_1, 2x_1, 1] \quad , \quad \mathbf{v}_2 = [x_2, 2x_2, 1]$$

we have

$$\mathbf{v}_1 - \mathbf{v}_2 = [x_1 - x_2, 2(x_1 - x_2), 0] \notin W$$

□

(e) $W = \{[2x_1, 3x_2, 4x_3, 5x_4] \mid x_i \in \mathbb{R}\}$ in $\mathbb{R}^4$

- Consider two arbitrary vectors in $\mathbb{R}^4$
$$\mathbf{x} = [x_1, x_2, x_3, x_4] \quad , \quad \mathbf{x}' = [x_1', x_2', x_3', x_4']$$
Then the vectors
$$\mathbf{v}_1 = [2x_1, 3x_2, 4x_3, 5x_4] \quad , \quad \mathbf{v}_2 = [2x_1', 3x_2', 4x_3', 5x_4']$$
will be in W. We have
$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= [2c_1x_1, 3c_1x_2, 4c_1x_3, 5c_1x_4] + [2c_1x_1', 3c_2x_2', 4c_2x_3', 5c_2x_4'] \\ &= [2(c_1x_1 + c_2x_1'), 3(c_1x_2 + c_2x_2'), 4(c_1x_3 + c_2x_3'), 5(c_4x_1 + c_2x_4')] \\ &= [2x_1'', 3x_2'', 4x_3'', 5x_4''] \end{aligned}$$
This vector belongs to W since
$$\mathbf{x}'' = [c_1x_1 + c_2x_1', c_1x_2 + c_2x_2', c_1x_3 + c_2x_3', c_4x_1 + c_2x_4'] \in \mathbb{R}^4$$
Since an arbitrary linear combinations of two vectors in W also lies in W, W is a subspace. □

2. (Problem 11 in text). Prove that the line $y = mx$ is a subspace of $\mathbb{R}^2$. (Hint: write the line as $W = \{[x, mx] \mid x \in \mathbb{R}\}$.)

- It suffices to show that an arbitrary linear combinations of two vectors in W also lies in W. Set
$$\mathbf{v}_1 = [x_1, mx_1] \quad , \quad \mathbf{v}_2 = [x_2, mx_2]$$
Then
$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 &= [c_1x_1 + c_2x_2, c_1mx_1 + c_2mx_2] \\ &= [(c_1x_1 + c_2x_2), m(c_1x_1 + c_2x_2)] \in W \end{aligned}$$
Hence, W is a subspace. □

3. (Problems 17,19 and 21 in text). Find a basis for the solution set of the following homogeneous linear systems.

(a)
$$\begin{aligned} 3x_1 + x_2 + x_3 &= 0 \\ 6x_1 + 2x_2 + 2x_3 &= 0 \\ -9x_1 - 3x_2 - 3x_3 &= 0 \end{aligned}$$

- This linear system corresponds to the following augmented matrices
$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 6 & 2 & 2 & 0 \\ -9 & -3 & -3 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + 3R_1]{R_3 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
The latter augmented matrix corresponds to
$$\begin{aligned} 3x_1 + x_2 + x_3 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$
Which is, effectively, one equation for three unknowns. Solving for x_1 in terms of x_2 and x_3 we obtain
$$x_1 = -\frac{1}{3}(x_2 + x_3)$$
So any vector of the form
$$\left[-\frac{1}{3}x_2 - \frac{1}{3}x_3, x_2, x_3\right] = x_2\left[-\frac{1}{3}, 1, 0\right] + x_3\left[-\frac{1}{3}, 0, 1\right]$$

will be a solution. We conclude that

$$\mathbf{e}_1 = \left[-\frac{1}{3}, 1, 0\right] \quad , \quad \mathbf{e}_2 = \left[-\frac{1}{3}, 0, 1\right]$$

will be a basis for the solution space. $\square$

(b)

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - 6x_2 + x_3 &= 0 \\ 3x_1 - 5x_2 + 2x_3 + x_4 &= 0 \\ 5x_1 - 4x_2 + 3x_3 + 2x_4 &= 0 \end{aligned}$$

- This system corresponds to the following augmented matrix

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 0 \\ 1 & -6 & 1 & 0 & 0 \\ 3 & -5 & 2 & 1 & 0 \\ 5 & -4 & 3 & 2 & 0 \end{array}\right]$$

Carrying out row reduction on this matrix yields

$$\begin{array}{l} R_2 \longrightarrow R_2 - \frac{1}{2}R_1 \\ R_3 \longrightarrow R_3 - \frac{3}{2}R_1 \\ R_4 \longrightarrow R_4 - \frac{5}{2}R_1 \end{array} \quad \Rightarrow \quad \left[\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 0 \\ 0 & -\frac{13}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{13}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{13}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\begin{array}{l} R_3 \longrightarrow R_3 - R_2 \\ R_4 \longrightarrow R_4 - R_2 \end{array} \quad \Rightarrow \quad \left[\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 0 \\ 0 & -\frac{13}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last augmented matrix corresponds to

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &= 0 \\ -\frac{13}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} x_2 &= \frac{1}{13}(x_3 + x_4) \\ x_1 &= -\frac{1}{2}(x_2 + x_3 + x_4) = \frac{1}{2}\left(\frac{14}{13}x_3 + \frac{14}{13}x_4\right) = \frac{7}{13}(x_3 + x_4) \end{aligned}$$

Hence, any vector of the form

$$\left[\frac{1}{13}(x_3 + x_4), \frac{7}{13}(x_3 + x_4), x_3, x_4\right] = x_3\left[\frac{1}{13}, \frac{7}{13}, 1, 0\right] + x_4\left[\frac{1}{13}, \frac{7}{13}, 0, 1\right]$$

will be a solution. Thus, the vectors

$$\mathbf{e}_1 = \left[\frac{1}{13}, \frac{7}{13}, 1, 0\right] \quad , \quad \mathbf{e}_2 = \left[\frac{1}{13}, \frac{7}{13}, 0, 1\right]$$

provide a basis for the solution space of the original set of equations. $\square$

(c)

$$\begin{aligned}
x_1 - x_2 + 6x_3 + x_4 - x_5 &= 0\\
3x_1 + 2x_2 - 3x_3 + 2x_4 + 5x_5 &= 0\\
4x_1 + 2x_2 - x_3 + 3x_4 - x_5 &= 0\\
3x_1 - 2x_2 + 14x_3 + x_4 - 8x_5 &= 0\\
2x_1 - x_2 + 8x_3 + 2x_4 - 7x_5 &= 0
\end{aligned}$$

- First we'll row-reduce the corresponding augmented matrix until it is in row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & -1 & 6 & 1 & -1 & 0\\ 3 & 2 & -3 & 2 & 5 & 0\\ 4 & 2 & -1 & 3 & -5 & 0\\ 3 & -2 & 14 & 1 & -8 & 0\\ 2 & -1 & 8 & 2 & -7 & 0 \end{array}\right]$$

$$\begin{array}{c} R_2 \rightarrow R_2 - 3R_1\\ R_3 \rightarrow R_3 - 4R_1\\ R_4 \rightarrow R_4 - 3R_1\\ R_5 \rightarrow R_5 - 2R_1 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccccc|c} 1 & -1 & 6 & 1 & -1 & 0\\ 0 & 5 & -21 & -1 & 8 & 0\\ 0 & 6 & -21 & -1 & -1 & 0\\ 0 & 1 & -4 & -2 & -5 & 0\\ 0 & 1 & -4 & 0 & -5 & 0 \end{array}\right]$$

$$\begin{array}{c} R_2 \rightarrow R_2 - 5R_5\\ R_3 \rightarrow R_3 - 6R_5\\ R_4 \rightarrow R_4 - R_5 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccccc|c} 1 & -1 & 6 & 1 & -1 & 0\\ 0 & 0 & -1 & -1 & 33 & 0\\ 0 & 0 & -1 & -1 & 29 & 0\\ 0 & 0 & 0 & -2 & 0 & 0\\ 0 & 1 & -4 & 0 & -5 & 0 \end{array}\right]$$

$$\begin{array}{c} R_2 \longleftrightarrow R_5\\ R_3 \rightarrow R_3 - 6R_5 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccccc|c} 1 & -1 & 6 & 1 & -1 & 0\\ 0 & 1 & -4 & 0 & -5 & 0\\ 0 & 0 & -1 & -1 & 29 & 0\\ 0 & 0 & 0 & -2 & 0 & 0\\ 0 & 0 & -1 & -1 & 33 & 0 \end{array}\right]$$

$$\begin{array}{c} R_2 \longleftrightarrow R_5\\ R_3 \rightarrow -R_3\\ R_4 \rightarrow -\frac{1}{2}R_4\\ R_5 \rightarrow (1/4)(R_5 - R_3) \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccccc|c} 1 & -1 & 6 & 1 & -1 & 0\\ 0 & 1 & -4 & 0 & -5 & 0\\ 0 & 0 & 1 & 1 & -29 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 \rightarrow R_1 + R_2\\ R_3 \rightarrow -R_3\\ R_4 \rightarrow -\frac{1}{2}R_4\\ R_5 \rightarrow (1/4)(R_5 - R_3) \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccccc|c} 1 & -1 & 6 & 1 & -1 & 0\\ 0 & 1 & -4 & 0 & -5 & 0\\ 0 & 0 & 1 & 1 & -29 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

This last augmented matrix corresponds to the following linear system:

$$\begin{aligned}
x_5 &= 0\\
x_4 &= 0\\
x_3 &= -x_4 + 29x_5 = 0\\
x_2 &= 4x_3 + 5x_5 = 0\\
x_1 &= x_2 - 6x_3 - x_4 + x_5 = 0
\end{aligned}$$

Thus, the only solution vector is $\mathbf{0} = [0,0,0,0,0]$. One can think of the zero vector as the basis vector for the solution subspace, however, it's probably better to think of the solution space as the zero vector. $\square$

4. (Problems 35 and 37 in text). Solve the following linear systems and express the solution set in a form that illustrates Theorem 1.18.

(a)

(1)
$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= -3 \\ 4x_1 + 4x_2 - x_4 &= 1 \end{aligned}$$

To solve the linear system we row reduce

$$\left[\begin{array}{cccc|c} 2 & -1 & 3 & 0 & -3 \\ 4 & 4 & 0 & -1 & 1 \end{array}\right]$$

$$R_2 \to (1/6))\left(R_2 - 2R_1\right) \quad \Rightarrow \quad \left[\begin{array}{cccc|c} 2 & -1 & 3 & 0 & -3 \\ 0 & 1 & -1 & -\frac{1}{6} & \frac{7}{6} \end{array}\right]$$

$$R_1 \to \frac{1}{2}R_1 + \frac{1}{2}R_2 \quad \Rightarrow \quad \left[\begin{array}{cccc|c} 1 & 0 & 1 & -\frac{1}{12} & -\frac{11}{12} \\ 0 & 1 & -1 & -\frac{1}{6} & \frac{7}{6} \end{array}\right]$$

From which we can infer

$$\begin{aligned} x_1 + x_3 - \frac{1}{12}x_4 &= -\frac{11}{12} \\ x_2 - x_3 - \frac{1}{6}x_4 &= \frac{7}{6} \end{aligned}$$

or

$$\begin{aligned} x_1 &= -x_3 + \frac{1}{12}x_4 - \frac{11}{12} \\ x_2 &= x_3 + \frac{1}{6}x_4 + \frac{7}{6} \end{aligned}$$

So a solution vector will have the form

$$\left[-x_3 + \frac{1}{12}x_4 - \frac{11}{12}, x_3 + \frac{1}{6}x_4 + \frac{7}{6}, x_3, x_4\right] = \left[-\frac{11}{12}, \frac{7}{6}, 0, 0\right] + x_3\left[-1,1,1,0\right] + x_4\left[\frac{1}{12}, \frac{1}{6}, 0, 1\right]$$

Setting

$$\mathbf{p} = \left[-\frac{11}{12}, \frac{7}{6}, 0, 0\right]$$

and

$$\mathbf{h} = r\left[-1,1,1,0\right] + s\left[\frac{1}{12}, \frac{1}{6}, 0, 1\right] = \left[-r + \frac{s}{12}, r + \frac{s}{6}, r, s\right]$$

and noting that $\mathbf{h}$ satisfies

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 2\left(-r + \frac{s}{12}\right) - \left(r + \frac{s}{6}\right) + 3r = r\left(-2 - 1 + 3\right) + s\left(\frac{2}{12} - \frac{1}{6}\right) = 0 \\ 4x_1 + 4x_2 - x_4 &= 4\left(-r + \frac{s}{12}\right) + 4\left(r + \frac{s}{6}\right) - s = r\left(-4 + 4\right) + s\left(\frac{4}{12} + \frac{4}{6} - 1\right) = 0 \end{aligned}$$

We see that an arbitrary solution of (??) can be expressed as

$$\mathbf{x} = \mathbf{p} + \mathbf{h}$$

with $\mathbf{p}$ a particular solution of (??) and $\mathbf{h}$ a solution of the corresponding homogeneous system.

(b)

$$(2)\qquad \begin{aligned} 2x_1 + x_2 + 3x_3 &= 5 \\ x_1 - x_2 + 2x_3 + x_4 &= 0 \\ 4x_1 - x_2 + 7x_3 + 2x_4 &= 5 \\ -x_1 - 2x_2 - x_3 + x_4 &= -5 \end{aligned}$$

The augmented matrix corresponding to this linear system is

$$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 0 & 5 \\ 1 & -1 & 2 & 1 & 0 \\ 4 & -2 & 7 & 2 & 5 \\ -1 & -2 & -1 & 1 & -5 \end{array}\right]$$

This matrix can be row-reduced to the following form:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 10 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which is equivalent to the following linear system

$$\begin{aligned} x_1 - x_4 &= 10 \\ x_2 &= 0 \\ x_3 + 2x_4 &= -5 \\ 0 &= 0 \end{aligned}$$

or

$$\begin{aligned} x_1 &= 3x_4 + 10 \\ x_2 &= 0 \\ x_3 &= -2x_4 - 5 \end{aligned}$$

Every solution vector can thus be written in the form

$$\mathbf{x} = [3x_4 + 10, 0, -2x_4 - 5, x_4] = [10, 0, -5, 0] + x_4\,[3, 0, -2, 1] = \mathbf{p} + \mathbf{h}$$

where $\mathbf{p}$ is a particular solution of (??) (the solution with $x_4 = 0$), and

$$\mathbf{h} = [3r, 0, -2r, 1]$$

is a solution of the corresponding homogeneous system.

5. (Problem 38 in text). Mark each of the following statements True or False.

a. A linear system with fewer equations than unknowns has an infinite number of solutions.

- False. If the system is inconsistent, it won't have any solutions. □

b. A consistent linear system with fewer equations than unknowns has an infinite number of solutions.

- True. □

c. If a square linear system $\mathbf{Ax} = \mathbf{b}$ has a solution for every choice of column vector $\mathbf{b}$, then the solution is unique for each choice of $\mathbf{b}$.

- True. This follows from Theorems 1.12 and 1.16. □

d. If a square system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, then $\mathbf{Ax} = \mathbf{b}$ has a unique solution for every column vector $\mathbf{b}$ with the appropriate number of components.

- True. This follows from Corollary 2 and Theorem 1.16 $\square$

e. If a linear system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, then $\mathbf{Ax} = \mathbf{b}$ has a unique solution for every column vector $\mathbf{b}$ with the appropriate number of components.

- False. Consider what happens when the matrix $\mathbf{A}$ is of the form

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then the corresponding system of equations

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

has a unique solution. Now consider a non-homogeneous linear system with the same matrix $\mathbf{A}$; say

$$\mathbf{Ax} = \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The corresponding augmented matrix will be

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 \end{array}\right]$$

and so we'll have an inconsistent set of equations whenever $b_3 \neq 0$. $\square$

f. The sum of two solution vectors of any linear system is also a solution vector of the system.

- False. Consider the following linear system:

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_3 &= 2 \end{aligned}$$

and the following two solution vectors

$$\mathbf{v}_1 = [1,1,0] \quad , \quad \mathbf{v}_2 = [1,0,1]$$

If we set

$$\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 = [2,1,1]$$

then, $\mathbf{v}_3$ will not be a solution vector. $\square$

g. The sum of two solution vectors of any homogeneous linear system is also a solution vector of the system.

- True. $\square$

h. A scalar multiple of a solution vector of any homogeneous linear system is also a solution vector of the system.

- True. The solution space of a homogeneous linear system is a subspace: therefore it's closed under scalar multiplication. $\square$

i. Every line in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ generated by a single vector.

- False. The line $\left\{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = (1,1) + (1,-1)t \quad ; \quad t \in \mathbb{R}\right\}$ is not a subspace. $\square$

j. Every line in $\mathbb{R}^2$ through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ generated by a single vector.

- True. $\square$