Math 3013
Problem Set 3

Problems from §1.5 (pg. 84-85 of text): 1,3,5,7,9,11,13,16

1. (Problems 1,3,5 and 7 in text.) Find the inverses of the following matrices, if they exist; and express each invertible matrix as a product of elementary matrices.

(a)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

-

$$\begin{aligned}[\mathbf{A} \mid \mathbf{I}] &= \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \\ R_1 &\rightarrow R_1 - R_2 \quad \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right] = [\mathbf{I} \mid \mathbf{A}^{-1}]\end{aligned}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

A is an elementary matrix (it is the elementary matrix corresponding to replacing the first row by its sum with the second row). □

(b)

$$\begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$$

-

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 4 & 8 & 0 & 1 \end{array}\right]$$

$$R_2 \rightarrow R_2 - \frac{4}{3}R_2 \quad \Rightarrow \left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{3} \end{array}\right]$$

This matrix does not have an inverse (it's row-echelon form has zero entries along the diagonal). □

(c)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

-

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

If we perform the following the elementary row operations

$$\begin{aligned} RowOp_1 &: R_1 \rightarrow R_1 + R_3 \\ RowOp_2 &: R_2 \rightarrow R_2 + R_3 \\ RowOp_3 &: R_3 \rightarrow -R_3 \end{aligned}$$

We convert the augmented matrix $[\mathbf{A} \mid \mathbf{I}]$ to the form

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] = \left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$$

So

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Our next task is to express $\mathbf{A}$ as a product to elementary matrices. Let $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ be the elementary matrices corresponding, respectively, to the elementary row operations $RowOp_1$, $RowOp_2$, and $RowOp_3$:

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Since the inverse of $\mathbf{A}$ was created by applying these same row operations to the identity matrix we have

$$\begin{aligned} \mathbf{A}^{-1} &= \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{I} \\ &= \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1 \end{aligned}$$

But then

$$\mathbf{A} = \left(\mathbf{A}^{-1}\right)^{-1} = \left(\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\right)^{-1} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}$$

Since the inverse of an elementary matrix is another (or perhaps the same) elementary matrix, The equation above tells us how to express $\mathbf{A}$ as a product of elementary matrices. All we have do to is figure out the inverses of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$. Now the inverse of an elementary matrix $\mathbf{E}$ is just the elementary matrix that *undoes* the row operation corresponding to $\mathbf{E}$. So

$$RowOp_1 \quad : \quad R_1 \to R_1 + R_3 \text{ is undone by } R_1 \to R_1 - R_3 \quad \Rightarrow \quad \mathbf{E}_1^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_2 \quad : \quad R_2 \to R_2 + R_3 \text{ is undone by } R_2 \to R_2 - R_3 \quad \Rightarrow \quad \mathbf{E}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_3 \quad : \quad R_3 \to -R_3 \text{ is undone by } R_3 \to -R_3 \quad \Rightarrow \quad \mathbf{E}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Hence

$$\mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\square$

(d)

$$\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 3 & 2 & 5 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

- We shall procede as in part (c).

$$
\begin{aligned}
RowOp_1 &: \quad R_2 \to R_2 - \frac{3}{2}R_1 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \\
RowOp_2 &: \quad R_3 \to R_3 - 2R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -1 & -3 & 2 & 1 \end{array}\right] \\
RowOp_3 &: \quad R_3 \to -R_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & -1 & 3 & -2 & -1 \end{array}\right] \\
RowOp_4 &: \quad R_2 \to 2R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right] \\
RowOp_5 &: \quad R_1 \to \frac{1}{2}R_1 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -2 & -3 & 2 & 0 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right] \\
RowOp_6 &: \quad R_2 \to R_2 + 2R_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right] \\
RowOp_7 &: \quad R_1 \to R_1 - \frac{1}{2}R_2 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 1 & 1 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right] \\
RowOp_8 &: \quad R_1 \to R_1 - 2R_3 \quad \Rightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 5 & 3 \\ 0 & 1 & 0 & 3 & -2 & -2 \\ 0 & 0 & 1 & 3 & -2 & -1 \end{array}\right]
\end{aligned}
$$

So

$$
\mathbf{A}^{-1} = \begin{bmatrix} -7 & 5 & 3 \\ 3 & -2 & -2 \\ 3 & -2 & -1 \end{bmatrix}
$$

To express $\mathbf{A}$ as a product of elementary matrices we procede as in part (c): we write

$$
\mathbf{A} = (\mathbf{E}_8\mathbf{E}_7\mathbf{E}_6\mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1)^{-1} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1}\mathbf{E}_4^{-1}\mathbf{E}_5^{-1}\mathbf{E}_6^{-1}\mathbf{E}_7^{-1}\mathbf{E}_8^{-1}
$$

where the $\mathbf{E}_1, \ldots, \mathbf{E}_8$ are the elementary matrices corresponding to the elementary row operations $RowOp_1, \ldots, RowOp_8$, and the $\mathbf{E}_1^{-1}, \ldots, \mathbf{E}_8^{-1}$ are their matrix inverses. We have

$$RowOp_1 \quad : \quad R_2 \to R_2 - \frac{3}{2}R_1 \quad \Rightarrow \quad \mathbf{E}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_2 \quad : \quad R_3 \to R_3 - 2R_2 \quad \Rightarrow \quad \mathbf{E}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$RowOp_3 \quad : \quad R_3 \to -R_3 \quad \Rightarrow \quad \mathbf{E}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$RowOp_4 \quad : \quad R_2 \to 2R_2 \quad \Rightarrow \quad \mathbf{E}_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_5 \quad : \quad R_1 \to \frac{1}{2}R_1 \quad \Rightarrow \quad \mathbf{E}_5^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_6 \quad : \quad R_2 \to R_2 + 2R_3 \quad \Rightarrow \quad \mathbf{E}_6^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_7 \quad : \quad R_1 \to R_1 - \frac{1}{2}R_2 \quad \Rightarrow \quad \mathbf{E}_7^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RowOp_8 \quad : \quad R_1 \to R_1 - 2R_3 \quad \Rightarrow \quad \mathbf{E}_8^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$\mathbf{E}_1^{-1} =$$

$\square$

2. (Problem 9 in text). Find the inverse of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

- Setting

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

and applying the following row operations

$$\begin{aligned} R_2 &\rightarrow -R_2 \\ R_3 &\rightarrow \frac{1}{2}R_3 \\ R_4 &\rightarrow \frac{1}{3}R_4 \\ R_5 &\rightarrow \frac{1}{4}R_5 \\ R_6 &\rightarrow \frac{1}{5}R_6 \end{aligned}$$

We obtain

$$\left[\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \end{array}\right] = \left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$$

So

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$\square$

3. (Problem 11 in text). Determine whether the span of the column vectors of the given matrix is $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 1 & 0 & -1 & 2 \\ -3 & 0 & 0 & -1 \end{bmatrix}$$

- According to the Theorem 1.12 in the text, all we have do is show that this matrix is row equivalent to the identity matrix. After carrying out the following two row operations

$$\begin{aligned} R_3 &\rightarrow R_3 - R_1 \\ R_4 &\rightarrow R_4 + 3R_1 \end{aligned}$$

we obtain

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 3 & -4 \end{bmatrix}$$

If we now carry out

$$R_4 \rightarrow R_4 + \frac{3}{2}R_3$$

we obtain

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & -1 & -3 & 4 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

This matrix is clearly row-equivalent to the identity matrix (it's in row-echelon form with no zero entries along the diagonal). Hence the span of the column vectors of the original matrix must be $\mathbb{R}^4$. $\square$

4. (Problem 13 in text). Show that the following matrix is invertible and find its inverse.

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 5 & -7 \end{bmatrix}$$

- Setting

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 5 & -7 & 0 & 1 \end{array}\right]$$

and carrying out row reduction

$$R_2 \rightarrow R_2 - \frac{5}{2}R_1 \quad \Rightarrow \quad \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right]$$

$$R_2 \rightarrow 2R_2 \quad \Rightarrow \quad \left[\begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

$$R_1 \rightarrow R_1 + 3R_2 \quad \Rightarrow \quad \left[\begin{array}{cc|cc} 2 & 0 & -14 & 6 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

$$R_1 \rightarrow \frac{1}{2}R_1 \quad \Rightarrow \quad \left[\begin{array}{cc|cc} 1 & 0 & -7 & 3 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

So

$$\mathbf{A}^{-1} = \begin{bmatrix} -7 & 3 \\ -5 & 2 \end{bmatrix}$$

$\square$

5. (Problem 16 in text). Let

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

If possible, find a matrix $\mathbf{C}$ such that

$$\mathbf{AC} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}$$

- Multiplying (from the left) both sides of the second equation by $\mathbf{A}^{-1}$ yields

$$\mathbf{A}^{-1}(\mathbf{AC}) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}$$

Now

$$\mathbf{A}^{-1}(\mathbf{AC}) = \left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{C} = \mathbf{IC} = \mathbf{C}$$

and

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 4 & 4 \\ 12 & 11 \end{bmatrix}$$

So

$$\mathbf{C} = \begin{bmatrix} 5 & 5 \\ 4 & 4 \\ 12 & 11 \end{bmatrix}$$

$\square$