Math 3013
Homework Set 1

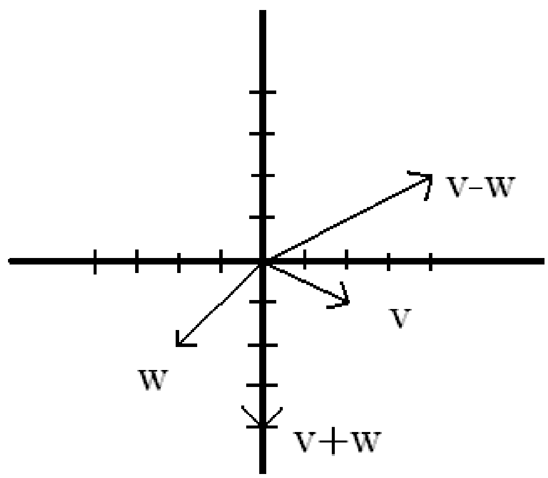
Problems from §1.1 (pg. 15 - 17 of text): 1,9,31,35

Problems from §1.2 (pg. 31 - 33 of text): 1,3,22,23,25,27,33

Problems from §1.3 (pg. 46 - 48 of text): 1,3,7,9,11,13,14,19,21

1. (Problem 1.1.1 in text). Let $\mathbf{v} = [2, -1]$ and $\mathbf{w} = [-2, -3]$. Compute $\mathbf{v} + \mathbf{w}$, $\mathbf{v} - \mathbf{w}$ and then draw coordinate axes and sketch, using your answers the vectors $\mathbf{v}, \mathbf{w}, \mathbf{v} + \mathbf{w}$, and $\mathbf{v} - \mathbf{w}$.

- $$\begin{aligned} \mathbf{v} + \mathbf{w} &= [2 + (-2), (-1) + (-3)] = [0, -4] \\ \mathbf{v} + \mathbf{w} &= [2 - (-2), (-1) - (-3)] = [4, 2] \end{aligned}$$



$\square$

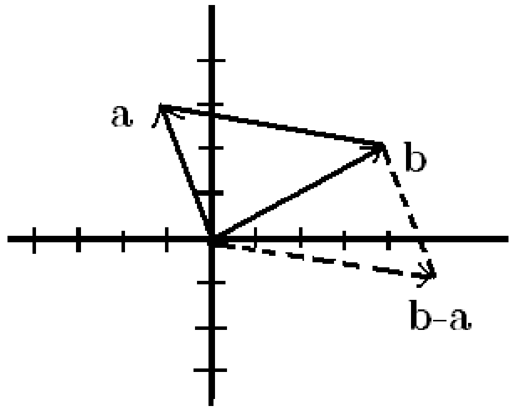
2. (Problem 1.1.9. in text). Let $\mathbf{u} = [1, 2, 1, 0]$, $\mathbf{v} = [-2, 0, 1, 6]$ and $\mathbf{w} = [3, -5, 1, -2]$. Compute $\mathbf{u} - 2\mathbf{v} + 4\mathbf{w}$.

- $$\begin{aligned} \mathbf{u} - 2\mathbf{v} + \mathbf{w} &= [1, 2, 1, 0] - 2[-2, 0, 1, 6] + 4[3, -5, 1, -2] \\ &= [1, 2, 1, 0] + [4, 0, -2, -12] + [12, -20, 4, -8] \\ &= [1 + 4 + 12, 2 + 0 - 20, 1 - 2 + 4, 0 - 12 - 8] \\ &= [17, -18, 3, -20] \end{aligned}$$

$\square$

3. (Problem 1.1.31 in text). Find the vector which, when translated, represents geometrically an arrow reaching from the point $(-1, 3)$ to the point $(4, 2)$ in $\mathbb{R}^2$.

- Set $\mathbf{a} = (\mathbf{-1}, \mathbf{3})$ and $\mathbf{b} = (4, 2)$. Then we have the following picture



The desired vector is a parallel transport of the vector $\mathbf{b}-\mathbf{a}=[4,2]-[-1,3]=[5,-1]$. $\square$

4. (Problems 1.2.1 and 1.2.3 in text). Let $\mathbf{u}=[-1,3,4]$ and $\mathbf{v}=[2,1,-1]$. Compute $\|-\mathbf{u}\|$ and $\|\mathbf{v}+\mathbf{u}\|$.

-
$$\begin{aligned} \|-\mathbf{u}\| &= \sqrt{\mathbf{u}\cdot\mathbf{u}}=\sqrt{(-1)^2+(3^2)+(4)^2}=\sqrt{1+9+16}=\sqrt{26} \\ \|\mathbf{v}+\mathbf{u}\| &= \sqrt{(\mathbf{v}+\mathbf{u})\cdot(\mathbf{v}+\mathbf{u})}=\sqrt{(-1+2)^2+(3+1)^2+(4-1)^2}=\sqrt{1+16+9}=\sqrt{26} \end{aligned}$$

$\square$

5. (Problem 1.2.22 in text). Compute the angle between $[1,-1,2,3,0,4]$ and $[7,0,1,3,2,4]$ in $\mathbb{R}^6$.

- Let $\mathbf{a}=[1,-1,2,3,0,4]$ and $\mathbf{b}=[7,0,1,3,2,4]$. From the geometric interpretation of the dot product, we have

$$\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\,\|\mathbf{b}\|\cos(\theta)$$

or

$$\cos(\theta)=\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$$

or

$$\theta=\cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right)$$

Now

$$\begin{aligned} \|\mathbf{a}\| &= \sqrt{(1)^2+(-1)^2+(2)^2+(3)^2+(0)^2+(4)^2}=\sqrt{31} \\ \|\mathbf{b}\| &= \sqrt{(7)^2+(0)^2+(1)^2+(3)^2+(2)^2+(4)^2}=\sqrt{79} \end{aligned}$$

and

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= (1)(7)+(-1)(0)+(2)(1)+(3)(3)+(0)(2)+(4)(4) \\ &= 7+0+2+9+16 \\ &= 34 \end{aligned}$$

:So

$$\theta=\cos^{-1}\left(\frac{34}{\sqrt{31}\sqrt{79}}\right)=.81338 \text{ radians}=46.603 \text{ degrees}$$

$\square$

6. (Problems 1.2.25 and 1.2.27 in text). Classify the vectors as parallel, perpendicular, or neither. If they are parallel, state whether they have the same or opposite directions.

(a) $[-1, 4]$ and $[8, 2]$.

- We have
$$[-1, 4] \cdot [8, 2] = -8 + 8 = 0$$
so the vectors must be perpendicular. $\square$

(b) $[3, 2, 1]$ and $[-9, -6, -3]$.

- We have
$$[-9, -6, -3] = -3[3, 2, 1]$$
so the two vectors are parallel. Since the scalar factor is negative, they point in the opposite directions. $\square$

7. (Problem 1.2.22 in text). Find the distance between the points $(2, -1, 3)$ and $(4, 1, -2)$ in $\mathbb{R}^3$.

- Let $\mathbf{a} = (2, -1, 3)$ and $\mathbf{b} = (4, 1, -2)$. The vector representing the displacement between the points $\mathbf{a}$ and $\mathbf{b}$ is
$$\mathbf{b} - \mathbf{a} = (4 - 2, 1 - (-1), -2 - 3) = (2, 2, -5)$$
And the length of this vector is the distance from $\mathbf{a}$ to $\mathbf{b}$. Thus,
$$\text{distance} = \|\mathbf{b} - \mathbf{a}\| = \sqrt{(2)^2 + (2)^2 + (-5)^2} = \sqrt{33}$$
$\square$

8. (Problems 1.3.1, 1.3.3, 1.3.7, 1.3.11,1.3.13, and 1.3.14 in text). Let
$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 2 & -1 \\ 0 & 6 \\ -3 & 2 \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix}$$

(a) $3\mathbf{A}$

- $$3\mathbf{A} = 3 \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 3 & 9 \\ 12 & 0 & -3 \end{bmatrix}$$
$\square$

(b)$\mathbf{A} + \mathbf{B}$

- $$\mathbf{A} + \mathbf{B} = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -2+4 & 1+1 & 3+-2 \\ 4+5 & 0+-1 & -1+3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 9 & -1 & 2 \end{bmatrix}$$
$\square$

(c) $\mathbf{AB}$

- $$\mathbf{AB} = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix}$$

  Which is undefined since the number of columns in the first matrix is not the same as the number of rows in the second matrix. □

(d) $\mathbf{A}^2$

- $$\mathbf{A}^2 = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}$$

  Which is undefined since the number of columns in the first matrix is not the same as the number of rows in the second matrix. □

(e) $(2\mathbf{A} - \mathbf{B})\,\mathbf{D}$

- $$\begin{aligned}
(2\mathbf{A} - \mathbf{B})\,\mathbf{D} &= \left(2\begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix}\right)\begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix} \\
&= \left(\begin{bmatrix} -4 & 2 & 6 \\ 8 & 0 & -2 \end{bmatrix} + \begin{bmatrix} -4 & -1 & 2 \\ -5 & 1 & -3 \end{bmatrix}\right)\begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix} \\
&= \left(\begin{bmatrix} -8 & 1 & 8 \\ 3 & 1 & -5 \end{bmatrix}\right)\begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix} \\
&= \begin{bmatrix} (-8,1,8)\cdot(-4,3,-1) & (-8,1,8)\cdot(2,5,-3) \\ (3,1,-5)\cdot(-4,3,-1) & (3,1,-5)\cdot(2,5,-3) \end{bmatrix} \\
&= \begin{bmatrix} (-8)(-4)+(1)(3)+(8)(-1) & (-8)(2)+(1)(5)+(8)(-2) \\ (3)(-4)+(1)(3)+(-5)(-1) & (3)(2)+(1)(5)+(-5)(-2) \end{bmatrix} \\
&= \begin{bmatrix} 27 & -27 \\ -4 & 21 \end{bmatrix}
\end{aligned}$$

  □

(f) $\mathbf{ADB}$

- $$\begin{aligned}
\mathbf{ADB} &= \mathbf{A}(\mathbf{DB}) = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} \left(\begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix} \right) \\
&= \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} (-4,2)\cdot(4,5) & (-4,2)\cdot(1,-1) & (-4,2)\cdot(-2,3) \\ (3,5)\cdot(4,5) & (3,5)\cdot(1,-1) & (3,5)\cdot(-2,3) \\ (-1,-3)\cdot(4,5) & (-1,-3)\cdot(1,-1) & (-1,-3)\cdot(-2,3) \end{bmatrix} \\
&= \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} -16+10 & -4-2 & 8+6 \\ 12+25 & 3-5 & -6+15 \\ -4-15 & -1+3 & 2-9 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} -6 & -6 & 14 \\ 37 & -2 & 9 \\ -19 & 2 & -7 \end{bmatrix} \\
&= \begin{bmatrix} (-2,1,3)\cdot(-6,37,-19) & (-2,1,3)\cdot(-6,-2,2) & (-2,1,3)\cdot(14,9,-7) \\ (4,0,-1)\cdot(-6,37,-19) & (4,0,-1)\cdot(-6,-2,2) & (4,0,-1)\cdot(14,9,-7) \end{bmatrix} \\
&= \begin{bmatrix} 12+37-57 & 12-2+6 & -28+9-21 \\ -24+0+19 & -24+0-2 & 56+0+7 \end{bmatrix} \\
&= \begin{bmatrix} -8 & 16 & -40 \\ 43 & -26 & 63 \end{bmatrix}
\end{aligned}$$

$\square$

9. (Problem 1.3.19 in text). Consider the row and column vectors

$$\mathbf{x} = [-2, 3, -1] \quad , \quad \mathbf{y} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

Compute the matrix products $\mathbf{xy}$ and $\mathbf{yx}$.

- $$\begin{aligned}
\mathbf{xy} &= [-2, 3, -1] \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = (-2,3,-1)\cdot(4,-1,3) = -14 \\
\mathbf{yx} &= \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} [-2, 3, -1] = (4,-1,3)\cdot(-2,3,-1) = -14
\end{aligned}$$

$\square$

10. (Problem 1.3.21 in text). Mark the following statements True or False.

a. If $\mathbf{A} = \mathbf{B}$, then $\mathbf{AC} = \mathbf{BC}$.

- True, and obvious. $\square$

b. If $\mathbf{AC} = \mathbf{BC}$, then $\mathbf{A} = \mathbf{B}$.

- False. Consider

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then $\mathbf{A} \neq \mathbf{B}$, but

$$\mathbf{AC} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{BC}$$

□

c. If $\mathbf{AB} = \mathbf{0}$, then $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

- False. Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1,0)\cdot(0,0) & (1,0)\cdot(0,1) \\ (0,0)\cdot(0,0) & (0,0)\cdot(0,1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

□

d. If $\mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{C}$, then $\mathbf{A} = \mathbf{B}$.

- True. Consider, the first equation, entry by entry.

$$\mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{C} \quad \Rightarrow \quad (\mathbf{A} + \mathbf{C})_{ij} = (\mathbf{B} + \mathbf{C})_{ij} \quad \Rightarrow \quad A_{ij} + C_{ij} = B_{ij} + C_{ij} \quad \Rightarrow \quad A_{ij} = B_{ij} \quad \Rightarrow \quad \mathbf{A} =$$

□

e. If $\mathbf{A}^2 = \mathbf{I}$, then $\mathbf{A} = \pm\mathbf{I}$.

- False. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq \pm\mathbf{I}$$

One has

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (0,1)\cdot(0,1) & (0,1)\cdot(1,0) \\ (1,0)\cdot(0,1) & (1,0)\cdot(1,0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

□

f. If $\mathbf{B} = \mathbf{A}^2$ and if $\mathbf{A}$ is an $n \times n$ matrix and symmetric, then $b_{ii} \geq 0$ for $i = 1, 2. \ldots, n$.

- True. If $\mathbf{B} = \mathbf{A}^2$, then

$$\begin{aligned} B_{ii} &= \left(\mathbf{A}^2\right)_{ii} \\ &\equiv \sum_{k=1}^{n} A_{ik} A_{ki} \\ &= \sum_{k=1}^{n} A_{ik} A_{ki} \quad , \quad \text{since } \mathbf{A} \text{ is symmetric} \\ &= \sum_{k=1}^{n} (A_{ik})^2 \\ &\geq 0 \quad , \quad \text{since it is a sum of non-negative numbers} \end{aligned}$$

□

g. If $\mathbf{AB} = \mathbf{C}$ and if two of the matrices are square, then so is the third.

- True. If $\mathbf{A}$ is an $m \times n$ matrix, then in order for the matrix multiplication on left to be defined, $\mathbf{B}$ must be a $n \times q$ matrix (for some q), and moreover, $\mathbf{C}$ must be a $m \times q$ matrix. Now, we have only three possibilities

$$\begin{array}{lllllll}
\mathbf{A} \text{ and } \mathbf{B} \text{ square} & \Rightarrow & m = n \text{ and } n = q & \Rightarrow & \mathbf{C} \text{ is } n \times n & \Rightarrow & \mathbf{C} \text{ is square} \\
\mathbf{A} \text{ and } \mathbf{C} \text{ square} & \Rightarrow & m = n \text{ and } n = q & \Rightarrow & \mathbf{B} \text{ is } n \times n & \Rightarrow & \mathbf{B} \text{ is square} \\
\mathbf{B} \text{ and } \mathbf{C} \text{ square} & \Rightarrow & q = n \text{ and } m = q & \Rightarrow & \mathbf{A} \text{ is } n \times n & \Rightarrow & \mathbf{A} \text{ is square}
\end{array}$$

$\square$

h. If $\mathbf{AB} = \mathbf{C}$ and if $\mathbf{C}$ is a column vector then so is $\mathbf{B}$.

- True. Suppose $\mathbf{A}$ is an $m \times n$ matrix, and $\mathbf{C}$ is a $q \times 1$ column vector then in order for the matrix multiplication on left to be defined, $\mathbf{B}$ must be a $1 \times q$ matrix; i.e., a column vector. $\square$

i. If $\mathbf{A}^2 = \mathbf{I}$, then $\mathbf{A}^n = \mathbf{I}$ for all integers $n \geq 2$.

- False. Consider the matrix $\mathbf{A}$ discussed in part (e). Evidently,

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \mathbf{A}\mathbf{I} = \mathbf{A} \neq \mathbf{I}$$

$\square$

j. If $\mathbf{A}^2 = \mathbf{I}$, then $\mathbf{A}^n = \mathbf{I}$ for all even integers $n \geq 2$.

- True. Suppose n is even. Write $n = 2k$. Then

$$\mathbf{A}^n = \mathbf{A}^{2k} = \left(\mathbf{A}^2\right)^k = (\mathbf{I})^k = \mathbf{I}$$

$\square$