Math 3013.004

SECOND EXAM

10:30 – 11:45 am, April 8, 1999

1.(10 pts) Determine if $S = \left\{ \left[x, \sqrt{x^2+y^2}, y \right] \mid x, y \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^3$.

- If S is a subspace it must be closed under vector addition. Consider $\mathbf{v}_1 = \left[1, \sqrt{2}, 1\right] \in S$ and $\mathbf{v}_2 = \left[-1, \sqrt{2}, -1\right] \in S$. Then

$$\mathbf{v}_1 + \mathbf{v}_2 = \left[0, 2\sqrt{2}, 0\right]$$

But, $\mathbf{v}_1 + \mathbf{v}_1 \notin S$, since $2\sqrt{2} \neq \sqrt{0^2+0^2} = 0$. So S is not a subspace.

2. Consider the following matrix: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 2 \end{bmatrix}$

(a) (10 pts) Find a basis for the column space of $\mathbf{A}$.

- First we row reduce $\mathbf{A}$ to a matrix $\mathbf{A}'$ in row-echelon form:

$$\mathbf{A} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{A}'$$

The pivots of $\mathbf{A}'$ occur in the first two columns; therefore, the first two columns of $\mathbf{A}$ will form a basis for the column space of $\mathbf{A}$:

$$ColSp\,(\mathbf{A}) = span\left([1,1,1], [2,3,1]\right)$$

(b) (10 pts) Find a basis for the row space of $\mathbf{A}$.

- The non-zero rows of $\mathbf{A}'$ will a basis for the row space of $\mathbf{A}$:

$$RowSp\,(\mathbf{A}) = span\left([0,2,2,1], [0,1,-1,-1]\right)$$

(c) (10 pts) Find a basis for the null space of $\mathbf{A}$.

- The null space of $\mathbf{A}$ will coincide with the solution space of $\mathbf{A}'\mathbf{x} = 0$, or

$$\Rightarrow \quad \begin{array}{l} x_1 + 2x_2 + 2x_3 + x_4 = 0 \\ x_2 - x_3 - x_4 = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} x_1 = -4x_3 - 3x_4 \\ x_2 = x_3 + x_4 \end{array}$$

$$\Rightarrow \quad \mathbf{z} = \begin{bmatrix} -4x_3 - 3x_4 \\ x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

So

$$Null(\mathbf{A}) = span\left(\begin{bmatrix} -4 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

(d) (5 pts) What is the rank of $\mathbf{A}$?

- $rank(\mathbf{A}) = \#$ basis vectors for $ColSp\,(\mathbf{A}) = \#$ basis vectors for $RowSp\,(\mathbf{A}) = 2$

3. Consider the following mapping: $T : \mathbb{R}^3 \to \mathbb{R}^2 : T([x_1, x_2, x_3]) = [x_1 + x_2, x_1 - x_3]$

(a) (5 pts) Show that T is a linear transformation.

- Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ be arbitrary vectors in $\mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Then

$$T(\lambda \mathbf{u}) = T(\lambda u_1, \lambda u_2, \lambda u_3) = [\lambda u_1 + \lambda u_2, \lambda u_1 - \lambda u_3] = \lambda [u_1 + u_2, u_1 - u_3] = \lambda T(\mathbf{u})$$

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(u_1 + v_1, u_2 + v_2, u_3 + v_3) = [u_1 + v_1 - (u_2 + v_2), u_1 + v_1 - (u_3 + v_3)] \\ &= [u_1 + u_2, u_1 - u_3] + [v_1 + v_2, v_1 - v_3] = T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

So T preserves both scalar multiplication and vector addition. Hence, it is a linear transformation.

(b) (5 pts) Find the matrix that represents T.

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$$\begin{aligned} T([1,0,0]) &= [1,1] \quad , \quad T([0,1,0]) = [1,0] \quad , \quad T([0,0,1]) = [0,-1] \\ &\Rightarrow \quad \mathbf{A}_T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \end{aligned}$$

(c) (10 pts) Find a basis for the range of T.

- The range of T will coincide with the column space of $\mathbf{A}_T$. $\mathbf{A}_T$ row reduces to

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

which has pivots in the first two columns. Hence, the first two columns of $\mathbf{A}$ will form a basis for the range of T.

$$range(T) = span([1,0], [1,-1]) = \mathbb{R}^2$$

4. (10 pts) Let $p_1 = 1 + x$, $p_2 = 1 + x + x^2$, $p_3 = 2x^2 - x - 1$. Find a basis for $span(p_1, p_2, p_3)$.

- The natural basis for polynomials of degree 2 is

$$\mathbf{e}_1 = 1 \quad , \quad \mathbf{e}_2 = x \quad , \quad \mathbf{e}_3 = x^2$$

In terms of this standard basis, we can represent p_1, p_2 and p_3 as

$$\begin{aligned} p_1 &= \mathbf{e}_1 + \mathbf{e}_2 \approx [1,1,0] \\ p_2 &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \approx [1,1,1] \\ p_3 &= -\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3 \approx [-1,-1,2] \end{aligned}$$

To find a basis for the span of the vectors $[1,1,0]$, $[1,1,1]$, and $[-1,-1,2]$ we arrange these vectors as the rows of a 3×3 matrix, row-reduce this matrix to row-echelon form, and identify the non-zero rows.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3\} \text{ is a basis}$$

$$\Rightarrow \quad \{1 + x, x^2\} \text{ is a basis}$$

5. Compute the determinants of the following matrices.

(a) (4 pts) $\begin{bmatrix} 2 & 3 & 0 \\ 4 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

- $$\begin{aligned} \det \begin{bmatrix} 2 & 3 & 0 \\ 4 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} &= 2\det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 3\det \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} + 0\det \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \\ &= 2\,(1-1) - 3\,(4-1) + 0 \\ &= -9 \end{aligned}$$

(b) (5 pts) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

- $$\begin{aligned} \det \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} &= \det \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix} = (1)(1)(1)(-3) \\ &= -3 \end{aligned}$$

6. Mark the following statements *True* or *False*. (3 pts each).

<u>*T*</u> (a) The number of linearly independent row vectors of a matrix is the same as the number of linearly independent column vectors.

<u>*F*</u> (b) The non-zero rows of a matrix **A** form a basis for the row space of **A**.

<u>*T*</u> (c) If an $n \times n$ matrix is invertible, then $rank\,(\mathbf{A}) = n$.

<u>*T*</u> (d) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then $span\,(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbb{R}^n$.

<u>*F*</u> (e) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a set of vectors in $\mathbb{R}^n$ such every $\mathbf{v} \in \mathbb{R}^n$ can be expressed as a linear combination of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ then is a basis for $\mathbb{R}^n$.

<u>*F*</u> (f) If the only solution of $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0$ is $c_1 = c_2 = \cdots = c_k = 0$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for $\mathbb{R}^n$.

<u>*T*</u> (g) A subspace of a vector space is also a vector space.