Math 3013.004

SOLUTIONS TO FIRST EXAM

1. (5 pts) Compute the angle between the vectors $\mathbf{u} = (1, 2, -1, 0)$ and $\mathbf{v} = (1, 1, 2, 1)$.

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$$\begin{aligned} \cos(\theta) &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} \\ \Rightarrow &\quad \theta = \cos^{-1}\left(\frac{(1 + 2 - 2 + 0)}{\sqrt{1^2 + 2^2 + (-1)^2 + 0^2}\sqrt{1^2 + 1^2 + 2^2 + 1^2}} \right) \\ &= \cos^{-1}\left(\frac{1}{\sqrt{6}\sqrt{7}} \right) = \cos^{-1}\left(\frac{1}{\sqrt{42}} \right) \end{aligned}$$

$\square$

2. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix}$$

Compute the following matrices.

(a) (5 pts) $\sqrt{2}\mathbf{A}$

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$$\mathbf{A} = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

$\square$

(b) (5 pts) $\mathbf{AB}$

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$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} (1,0,1)\cdot(1,2,3) & (1,0,1)\cdot(0,-1,1) \\ (0,1,0)\cdot(1,2,3) & (0,1,0)\cdot(0,-1,1) \\ (-1,-1,1)\cdot(1,2,3) & (-1,-1,1)\cdot(0,-1,1) \end{bmatrix} \\ &= \begin{bmatrix} 1+0+3 & 0+0+1 \\ 0+2+0 & 0-1+0 \\ -1-2+3 & 0+1+1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 2 & -1 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

$\square$

(c) (5 pts) $\mathbf{BA}$

- $\mathbf{BA}$ is undefined since the number of columns of $\mathbf{B}$ is not equal to the number of rows of $\mathbf{A}$

$\square$

(d) (5 pts) $\mathbf{A} + 2\mathbf{B}$

$\mathbf{A} + 2\mathbf{B}$ is undefined since $\mathbf{A}$ is 3×3 and $\mathbf{B}$ is 3×2

$\square$

3. Consider the following linear system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= -1 \\ 3x_1 + x_2 + 2x_3 &= 3 \\ x_1 + x_3 &= 4 \end{aligned}$$

(a) (5 pts) Write down the corresponding augmented matrix and reduce it to row-echelon form.

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$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & -1 \\ 3 & 1 & 2 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right] \xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & -3 \\ 0 & 4 & -4 & 0 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{4}R_3]{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - 3R} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

□

(b) (5 pts) Reduce the augmented matrix further to **reduced** row-echelon form.

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$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\substack{R_1 \to R_1 - R_3 \\ R_2 \to R_2 + R_3}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

□

(c) (5 pts) Write down the solution of the original linear system.

- There is no solution since the corresponding equations are inconsistent. To see this explicitly, note that the equation corresponding to the third row of the reduced row-echelon form of the augmented matrix is

$$0 = 0x_1 + 0x_2 + 0x_3 = 1$$

  which is clearly contradictory.

□

4. (10 pts) Compute the inverse of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

and verify that you have the correct inverse.

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$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -2 & -3 & -4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to -R_2]{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

$$\Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

To verify, one checks

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

□

5. (10 pts) Use the fact that

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 3 & 0 \\ 1 & -1 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

to solve

$$\begin{bmatrix} 1 & -1 & -1 \\ -2 & 3 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

- Multiplying both sides of the equation above by $\mathbf{A}^{-1}$ yields

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -2 & 3 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

The left hand side of this equation is $\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{x} = \mathbf{I}\mathbf{x} = \mathbf{x}$. The right hand side is

$$\begin{bmatrix} (0,1,3)\cdot(1,2,1) \\ (0,1,2)\cdot(1,2,1) \\ (-1,0,1)\cdot(1,2,1) \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

So we can conclude

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

□

6. (10 pts) Determine if the set $W = \{(x, y, z) \in \mathbb{R}^3 \mid y = x, z = 2x\}$ is a subspace of $\mathbb{R}^3$.

- Let $\mathbf{w}_1 = (x_1, x_1, 2x_1)$ and $\mathbf{w}_2 = (x_2, x_2, 2x_2)$ be arbitrary vectors in W. Then for any real number c

$$\begin{aligned} c\mathbf{w}_1 &= (cx_1, cx_1, 2cx_1) \in W \quad \text{(by taking } x = cx_1) \\ \mathbf{w}_1 + \mathbf{w}_2 &= (x_1 + x_2, x_1 + x_2, 2x_1 + 2x_2) = (x_1 + x_2, x_1 + x_2, 2x_1 + 2x_2) \in W \quad \text{(by taking } x = x_1 + x_2) \end{aligned}$$

So since W is closed under scalar multiplication and vector addition, W is a subspace of $\mathbf{R}^3$. □

7. (10 pts) Find a basis for the solution set of the following homogeneous linear system.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 + x_2 + 3x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

- The coefficient matrix corresponding to this homogeneous linear system is

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 \to -R_2]{R_1 \to R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The homogeneous linear system corresponding to the reduction of $\mathbf{A}$ to reduced row-echelon form is

$$\left.\begin{array}{c} x_1 + 5x_3 = 0 \\ x_2 - 2x_3 = 0 \\ 0 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x_1 = -5x_3 \\ x_2 = 2x_3 \end{array}\right.$$

Any solution vector can thus be written in the form

$$\mathbf{x} = \begin{bmatrix} -5x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

So

$$\begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$$

is a basis for the solution space ot the homogeneous linear system.

□

8. (20 pts) Mark each ot the following statements True or False. (Think carefully.)

<u>*T*</u> (a) If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are invertible $n \times n$ matrices, then $\mathbf{AC} = \mathbf{BC}$ implies $\mathbf{A} = \mathbf{B}$.
<u>*F*</u> (b) If $\mathbf{A}$ and $\mathbf{B}$ are invertible $n \times n$ matrices, then $\mathbf{AB} = \mathbf{BA}$ implies $\mathbf{B} = \mathbf{A}^{-1}$.

Consider

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Then

$$\mathbf{AB} = \mathbf{BA} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

but

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \neq \mathbf{B}$$

$\underline{\quad F \quad}$ (c) If a consistent linear system has more equations than unknowns, then there will be a unique solution.

Consider the following system

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \\ 3x + 3y &= 3 \end{aligned}$$

This is a system of three equations and two unknowns that has infinitely many solutions ($x = r, y = 1 - r$; $r \in \mathbb{R}$). Although this appears to be a trivial counter-example, you need to be aware of this there are also many situations where one has redundant equations, yet the redundancies are not so easy to spot.

$\underline{\quad T \quad}$ (d) If a square linear system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution, then every linear system of the form $\mathbf{Ax} = \mathbf{b}$ will have a unique solution.

$\underline{\quad F \quad}$ (e) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of a consistent linear system $\mathbf{Ax} = \mathbf{b}$, then so is $\mathbf{x}_1 + \mathbf{x}_2$.

This is a false generalization of the fact that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of a consistent linear system $\mathbf{Ax} = \mathbf{0}$, then so is $\mathbf{x}_1 + \mathbf{x}_2$

$\underline{\quad T \quad}$ (f) If $\mathbf{p}$ is a solution of $\mathbf{Ax} = \mathbf{b}$ then every other solution can be written as $\mathbf{x} = \mathbf{p} + \mathbf{h}$ where $\mathbf{h}$ is a solution of the corresponding homogeneous equation.

$\underline{\quad F \quad}$ (g) Every line in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.

A line is a subspace if and only if it passes through the origin.

$\underline{\quad T \quad}$ (h) If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in $\mathbf{R}^3$, then every vector in $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ can be represented as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

$\underline{\quad F \quad}$ (i) If every vector in a subspace W of $\mathbb{R}^4$ can be represented as a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{R}^4$, then $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for W.

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ must also be linearly independent if they are to form a basis for their span.

$\underline{\quad T \quad}$ (j) A square linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution if and only if $\mathbf{A}$ is row-equivalent to the identity matrix.