LECTURE 15

Eigenvalues and Eigenvectors

1. Motivating Examples

EXAMPLE 15.1. Consider the following linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$:

$$T([x, y, z]) = [x, y\cos(\theta) + z\sin(\theta), -y\sin(\theta) + z\cos(\theta)]$$

The corresponding matrix is

$$\mathbf{A}_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

This transformation corresponds to a rotation about the x-axis by θ.

QUESTION 15.2. *What vectors in $\mathbb{R}^3$ are unaffected by this transformation?*

This question is equivalent to solving the linear system

$$\mathbf{A}_T \mathbf{x} = \mathbf{x} \tag{15.1}$$

or

$$(\mathbf{A}_T - \mathbf{I})\,\mathbf{x} = \mathbf{0}$$

or

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(\theta) - 1 & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row-reducing the matrix on the left-hand side yields

$$\longrightarrow \begin{bmatrix} 0 & -\sin(\theta) & \cos(\theta) - 1 \\ 0 & 0 & -2\frac{\cos(\theta)-1}{\sin(\theta))} \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{c} x \text{ is arbitrary} \\ y = 0 \\ z = 0 \end{array}$$

In other words, the transformation T leaves the points along the x-axis unaffected, and **only** those points remain unaffected by the transformation.

EXAMPLE 15.3. Let $C^\infty(\mathbb{R})$ denote the vector space of smooth (infinitely differentiable) functions on the real line. Consider the transformation $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by

$$D[f] = \frac{df}{dx}$$

It is easy to see that this is a linear transformation (it preserves scalar multiplication and vector addition in $C^\infty(\mathbb{R})$). We can associate with D an infinite matrix (employing simple monomials in x, $\left\{1, x, x^2, x^3, \dots\right\}$

as a standard basis for $C^{\infty}(\mathbb{R})$):

$$
\begin{aligned}
D[1] &= 0 \\
D[x] &= 1 \\
D\left[x^2\right] &= 2x \\
D\left[x^3\right] &= 3x^2 \\
&\vdots \\
\Rightarrow \qquad \mathbf{A}_D &= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}
\end{aligned}
$$

Solution of a differential equation of the form

$$\frac{df}{dx} = \lambda f$$

is then equivalent to a linear system of the form

$$\mathbf{A}_D \mathbf{f} = \lambda \mathbf{f} \tag{15.2}$$

Note the similarity of this equation with (15.1).

These examples motivate the following definition.

DEFINITION 15.4. *Suppose* $\mathbf{A}$ *is an* $n \times n$ *matrix. An* ***eigenvector*** *of* $\mathbf{A}$ *is a non-zero vector* $\mathbf{v} \in \mathbb{R}^n$ *such that*

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \quad , \quad \lambda \in \mathbb{R} \tag{15.3}$$

A number λ *such that 15.3 is true is called an* ***eigenvalue*** *of* $\mathbf{A}$*.*

In terms of this new semantics, the first example shows that the vectors along the x-axis are eigenvectors of the linear transformation T with eigenvalues 1; and the second example shows that the solution of a differential equation can be thought of as a problem of finding an eigenvector for a linear transformation on the vector space of smooth functions on $\mathbb{R}$.

2. Computing Eigenvalues

For a given $n \times n$ matrix $\mathbf{A}$, the **eigenvalue problem** is the problem of finding the eigenvalues and eigenvectors of $\mathbf{A}$. Our abstract machinery now plays a crucial role.

Before finding a solution $\mathbf{v}$ of

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \quad , \quad \lambda \in \mathbb{R} \tag{15.4}$$

we first ask, *when do non-trivial solutions exist?*

First we rewrite (15.4) as

$$\mathbf{A}\mathbf{v} - \lambda \mathbf{v} = \mathbf{0}$$

or

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{v} = \mathbf{0}$$

Now recall that demanding that the existence of non-zero solutions to linear system

$$\mathbf{Bx} = \mathbf{0}$$

is equivalent to demanding that the null space of $\mathbf{B}$ is at least 1-dimensional, which is equivalent to saying that when $\mathbf{B}$ is row-reduced to row-echelon form there is at least one row full of zeros, which implies $\det(\mathbf{B}) = 0$.

Thus, we arrive at the following criteria for the existence of eigenvectors and eigenvalues.

ASSERTION 15.1. If non-trivial solutions of $\mathbf{Av} = \lambda\mathbf{v}$ exist then

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

EXAMPLE 15.5. Find the possible eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$

- If $\mathbf{A}$ is to have eigenvalues, then we must have

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} 3-\lambda & 2 \\ 2 & -\lambda \end{bmatrix} = (3-\lambda)(-\lambda) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

  Hence, the possible eigenvalues are 4 and 1.

This simple example actually serves as the prototype for the general case:

To find the eigenvalues of an $n \times n$ matrix $\mathbf{A}$:

FACT 15.6.
1. *Compute the determinant of the matrix* $\mathbf{H} = \mathbf{A} - \lambda\mathbf{I}$, *regarding* λ *as a symbolic variable.*
2. $\det(\mathbf{A} - \lambda\mathbf{I})$ *will be a polynomial* $P_{\mathbf{A}}(\lambda)$ *of degree* n *in* λ. *We shall call this polynomial the* ***characteristic polynomial*** *of* $\mathbf{A}$.
3. *Solve* $P_{\mathbf{A}}(\lambda) = 0$. *The roots of this equation will be the possible eigenvalues of* $\mathbf{A}$.

At this point it is worthwhile to recall the Fundamental Theorem of Algebra.

THEOREM 15.7. *Every polynomial* $p(\lambda)$ *of degree* n *has a unique factorization in terms of linear polynomials:*

$$p(\lambda) = c(\lambda - r_1)^{m_1}(\lambda - r_2)^{m_2} \cdots (\lambda - r_s)^{m_s}$$

where $r_1 \neq r_2 \neq \cdots \neq r_s$, *and* $\sum_{i=1}^{s} m_i = n$. *The numbers* r_i *are, in general, complex numbers and are called the* ***roots*** *of* $p(x)$*; they coincide precisely with the solutions of* $p(\lambda) = 0$. *The integers* m_i *are called the* ***multiplicities*** *of the corresponding roots.*

This theorem tells us that the eigenvalues of a matrix can be complex numbers (in fact, this is what one must expect in general), and that they may occur with some multiplicity. The multiplicity of an eigenvalue will be very relevant to the problem of diagonalizing matrices (which will take up in the next lecture).

EXAMPLE 15.8. Find the possible eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

- We have

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} -\lambda & 2 \\ -2 & -\lambda \end{bmatrix} = \lambda^2 + 4$$

  The characteristic polynomial is thus

$$P_{\mathbf{A}}(\lambda) = \lambda^2 + 4 = (\lambda + 2i)(\lambda - 2i)$$

  Thus, the possible eigenvalues are $\lambda = \pm 2i$.

EXAMPLE 15.9. Find the possible eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

- We have

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \det \begin{bmatrix} -\lambda & 0 & 1 \\ -2 & 1-\lambda & 1 \\ 2 & 0 & -1-\lambda \end{bmatrix} \\ &= (-\lambda)\det \begin{bmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix} - (0)\det \begin{bmatrix} -2 & 1 \\ 2 & -1-\lambda \end{bmatrix} + (1)\det \begin{bmatrix} -2 & 1-\lambda \\ 2 & 0 \end{bmatrix} \\ &= \lambda(1-\lambda)(1+\lambda) + 0 - (2)(1-\lambda) \\ &= 3\lambda - \lambda^3 - 2 \\ &= -(2+\lambda)(\lambda-1)^2 \end{aligned}$$

We thus see that $\mathbf{A}$ has two possible eigenvalues, $\lambda = -2$ and $\lambda = 1$. The eigenvalue $\lambda = 1$ occurs with multiplicity 2.

3. Computation of Eigenvectors

In finding the eigenvalues of a matrix, all we have accomplished is figuring out for what values of λ

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{15.5}$$

can have a solution. I will now show you how to find the corresponding eigenvectors $\mathbf{v}$.

This is very straight-forward given our recent discussion of homogeneous linear systems. For finding vectors $\mathbf{v}$ such that (15.5) is satisfied is equivalent to finding solutions of

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$$

Let's do a quick example.

EXAMPLE 15.10. Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$

- In the first example, we discovered that the possible eigenvalues for this matrix are $\lambda = 4, -1$.
  Let's consider find first the solution of

$$(\mathbf{A} - 4\mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{15.6}$$

(which should be an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $\lambda = 4$). We have

$$\mathbf{A} - 4\mathbf{I} = \begin{bmatrix} 3-4 & 2 \\ 2 & 0-4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$

This matrix row reduces to

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

And so the solutions of (15.6) coincide with solutions of

$$\begin{matrix} x_1 - 2x_2 = 0 \\ 0 = 0 \end{matrix} \quad \Rightarrow \quad x_1 = 2x_2 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad , \quad x_2 \in \mathbb{R}$$

Thus, any vector of the form

$$\mathbf{v} = r \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad , \quad r \in \mathbb{R}$$

will be an eigenvector of $\mathbf{A}$ with eigenvalue 4.

Now let's look for eigenvectors corresponding to the eigenvalue -1. We thus look for solutions of

$$(\mathbf{A} + \mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{15.7}$$

We have

$$\mathbf{A} + \mathbf{I} = \begin{bmatrix} 3+1 & 2 \\ 2 & 0+1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

which is row-equivalent to

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

So the solutions of (15.7) coincide with the solutions of

$$\begin{array}{c} 2x_1 + x_2 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad x_1 = -\frac{1}{2}x_2 \quad \Rightarrow \quad \mathbf{x} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

So the corresponding eigenvectors will be vectors of the form

$$\mathbf{v} = r \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Example 15.11. Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

- Recall from Example 15.9 that the characteristic polynomial for this matrix factorizes as

$$P_{\mathbf{A}}(\lambda) = -(2+\lambda)(\lambda-1)^2$$

and so we have two possible eigenvalues: $\lambda = -2$ with multiplicity 1, and $\lambda = 1$ with multiplicity 2.

Let's look for first for eigenvectors corresponding to the eigenvalue $\lambda = -2$. We have

$$\mathbf{A} - (-2)\mathbf{I} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix} \quad \underrightarrow{\text{row reduction}} \quad \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and so the solutions $\mathbf{x}$ of $(\mathbf{A} + 2\mathbf{I})\,\mathbf{x} = \mathbf{0}$ satisfy

$$\begin{array}{c} 2x_1 + x_3 = 0 \\ 3x_2 + 2x_3 = 0 \\ 0 = 0 \end{array} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} -\frac{1}{2}x_3 \\ -\frac{2}{3}x_3 \\ x_3 \end{bmatrix} \in span\left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\right)$$

So the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue -2 are scalar multiples of the vector $\left[-2, -\frac{2}{3}, 1\right]$.

We'll now repeat the calculation for the double root $\lambda = 1$. We have

$$\mathbf{A} - (1)\mathbf{I} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 1 \\ 2 & 0 & -2 \end{bmatrix} \quad \underrightarrow{\text{row reduction}} \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and so the solutions $\mathbf{x}$ of $(\mathbf{A}-\mathbf{I})\,\mathbf{x}=\mathbf{0}$ satisfy

$$\begin{array}{c} x_1 - x_3 = 0 \\ -x_3 = 0 \\ 0 = 0 \end{array} \qquad \Rightarrow \qquad \mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$$

So the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue 1 are scalar multiples of the vector $[0,1,0]$.