LECTURE 13

Determinants

1. Calculating the Area of a Parallelogram

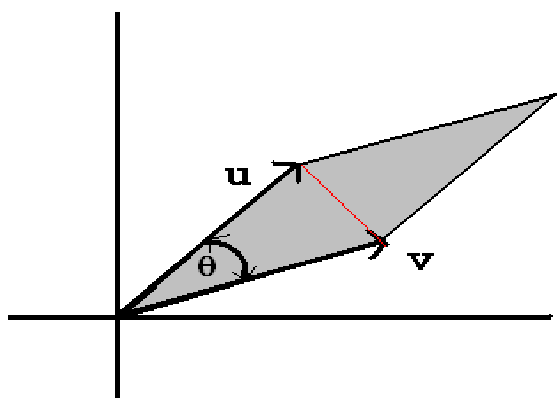
DEFINITION 13.1. *Let* $\mathbf{A}$ *be a* 2×2 *matrix.*

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

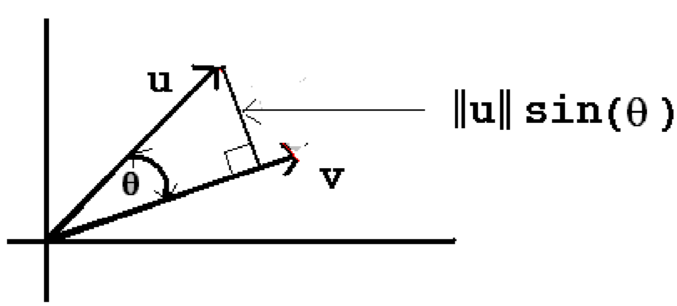
The determinant of $\mathbf{A}$ *is the number*

$$\det(\mathbf{A}) = ad - bc$$

Now consider the parallelogram formed from two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$.



The area of this parallelogram is evidently the sum of the areas the two isometric triangles obtained by bisecting the parallelogram along the line from the tip of $\mathbf{u}$ to the tip of $\mathbf{v}$. Now the height of a triangle formed from $\mathbf{u}$ and $\mathbf{v}$ will be given by $\|\mathbf{u}\| \sin(\theta)$.



The area of this triangle is given by a formula from high school geometry:

$$\begin{aligned}\text{area of a triangle} &= \frac{1}{2}(\text{length of base}) \times (\text{height of triangle}) \\ &= \frac{1}{2}\|\mathbf{v}\|\,\|\mathbf{u}\|\sin(\theta)\end{aligned}$$

Hence, the area for the parallelogram will be

$$\begin{aligned}\text{area of parallelogram} &= 2 \text{ times area of } \Delta_{\mathbf{uv}} \\ &= 2\cdot\frac{1}{2}\|\mathbf{v}\|\,\|\mathbf{u}\|\sin(\theta) \\ &= \|\mathbf{v}\|\,\|\mathbf{u}\|\sin(\theta) \\ &= \|\mathbf{v}\|\,\|\mathbf{u}\|\sqrt{1-\cos^2(\theta)}\end{aligned}$$

If we square both sides we have

$$\begin{aligned}area^2 &= \|\mathbf{v}\|^2\|\mathbf{u}\|^2\left(1-\cos^2(\theta)\right) \\ &= \|\mathbf{v}\|^2\|\mathbf{u}\|^2\left(1-\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|\,\|\mathbf{u}\|}\right)^2\right) \\ &= \|\mathbf{v}\|^2\|\mathbf{u}\|^2-(\mathbf{u}\cdot\mathbf{v})^2 \\ &= (\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{u})-(\mathbf{u}\cdot\mathbf{v})^2 \\ &= \left(v_1^2+v_2^2\right)\left(u_1^2+u_2^2\right)-(u_1v_1+u_2v_2)^2 \\ &= v_1^2u_1^2+v_1^2u_2^2+v_2^2u_1^2+v_2^2u_2^2-v_1^2u_1^2-2u_1v_1u_2v_2-v_2^2u_2^2 \\ &= v_1^2u_2^2+v_2^2u_1^2-2u_1v_1u_2v_2 \\ &= (u_1v_2-u_2v_1)^2\end{aligned}$$

So

$$area = |u_1v_2-u_2v_1|$$

Now let $\mathbf{A}$ be the 2×2 matrix formed by interpreting $\mathbf{u},\mathbf{v}\in\mathbb{R}^2$ as its columns:

$$\mathbf{A} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

Evidently,

$$|\det(\mathbf{A})| = |u_1v_2-u_2v_1| = \text{area of parallelogram formed from } \mathbf{u} \text{ and } \mathbf{v}$$

2. Calculating the Volume of a Parallelopiped

DEFINITION 13.2. *Let $\mathbf{A}$ be a 3×3 matrix.*

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant of $\mathbf{A}$ is the number

$$\begin{aligned}\det(\mathbf{A}) &= a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32} \\ &\quad -a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}\end{aligned}$$

A **parallelopiped** is the 3-dimensional analog a 2-dimensional parallelogram; these are constructed by regarding three 3-dimensional vectors and their translates as the edges of a solid body, and the parallelograms formed from pairs of these vectors as the sides. We shall not work out the geometry of this example but simply state the following fact:

FACT 13.3. *Let $P_{\mathbf{a},\mathbf{b},\mathbf{c}}$ be the parallelopiped associated with three vectors $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbb{R}^3$, and let $\mathbf{A}$ be the 3×3 matrix formed by regarding the vectors $\mathbf{a},\mathbf{b}$, and $\mathbf{c}$ as its column vectors. Then*

$$\text{volume of } P_{\mathbf{a},\mathbf{b},\mathbf{c}} = \left|\det(\mathbf{A})\right|$$

3. More General Determinants

The formula for the determinant of a general $n \times n$ will in general involve $n!$ separate terms; thus, the determinant for a 4×4 matrix will involve $4! = 24$ terms, and the determinant for a 5×5 matrix will involve $5! = 60$ terms! Rather than giving an explicit formula for the higher order determinants, we'll present an algorithmic construction.

DEFINITION 13.4. *Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix. The* ***minor matrix*** *corresponding to the entry a_{ij} is the $(n-1) \times (n-1)$ matrix formed from $\mathbf{A}$ by deleting the i^{th} row and j^{th} column of $\mathbf{A}$.*

DEFINITION 13.5. *The determinant of a 1×1 matrix is its sole entry. Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix. The* ***cofactor*** *C_{ij} of a_{ij} is $(-1)^{i+j}$ times the determinant of the minor matrix M_{ij} corresponding to a_{ij}.*

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

The determinant of $\mathbf{A}$ is

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$$

EXAMPLE 13.6. Calculate the determinant of the following matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

- We have

$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 \end{vmatrix} = (1)\begin{vmatrix} 0 & 1 & 2 \\ 1 & -1 & 2 \\ 1 & 1 & 0 \end{vmatrix} - (2)\begin{vmatrix} -1 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{vmatrix} + (0)\begin{vmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix} - (1)\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

Now

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & -1 & 2 \\ 1 & 1 & 0 \end{vmatrix} = (0)\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} - (1)\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} + (2)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 0 - (0-2) + 2(1+1) = 6$$

$$\begin{vmatrix} -1 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} - (1)\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} + (2)\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = -(-2) - (-2) + 2(1) = 6$$

$$\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - (0)\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -(1+1) + 0 + (-1) = -3$$

So

$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 \end{vmatrix} = (1)(6) - (2)(6) + 0 - (-3) = -3$$

4. Properties of Determinants

THEOREM 13.7.
1. *For any square matrix* $\mathbf{A}$, *we have* $\det\left(\mathbf{A}^T\right) = \det(\mathbf{A})$.
2. *If* $\mathbf{A}$ *is a square matrix and* $\mathbf{A}'$ *is a matrix obtained from* $\mathbf{A}$ *by interchanging two of its rows then* $\det(\mathbf{A}') = -\det(\mathbf{A})$.
3. *If two rows of a square matrix* $\mathbf{A}$ *are identical then* $\det(\mathbf{A}) = 0$.
4. *If* $\mathbf{A}$ *is a square matrix and* $\mathbf{A}'$ *is a matrix obtained from* $\mathbf{A}$ *by multiplying one of its rows by a scalar* r *then* $\det(\mathbf{A}') = r\det(\mathbf{A})$
5. *If* $\mathbf{A}$ *is a square matrix and* $\mathbf{A}'$ *is a matrix obtained from* $\mathbf{A}$ *by adding a scalar multiple of one row to another, then* $\det(\mathbf{A}') = \det(\mathbf{A})$.

The above theorem tells us what effect each of the elementary row operations has on the determinant of a matrix. We also have

LEMMA 13.8. *If* $\mathbf{A}$ *is an upper triangular matrix then* $\det(\mathbf{A})$ *is equal to the products of the entries along the diagonal.*

We won't prove this theorem in general. However, the basic idea underlying this theorem is easily demonstrated by a simple example.

EXAMPLE 13.9. Compute the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

We have

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{vmatrix} &= (1)\begin{vmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{vmatrix} - (2)\begin{vmatrix} 0 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{vmatrix} + (3)\begin{vmatrix} 0 & 5 & 7 \\ 0 & 0 & 9 \\ 0 & 0 & 10 \end{vmatrix} - (4)\begin{vmatrix} 0 & 5 & 6 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{vmatrix} \\
&= (1)\left((5)\begin{vmatrix} 8 & 9 \\ 0 & 10 \end{vmatrix} - (6)\begin{vmatrix} 0 & 9 \\ 0 & 10 \end{vmatrix} + (7)\begin{vmatrix} 0 & 8 \\ 0 & 0 \end{vmatrix}\right) \\
&\quad -(2)\left((0)\begin{vmatrix} 8 & 9 \\ 0 & 10 \end{vmatrix} - (6)\begin{vmatrix} 0 & 9 \\ 0 & 10 \end{vmatrix} + (7)\begin{vmatrix} 0 & 8 \\ 0 & 0 \end{vmatrix}\right) \\
&\quad +(3)\left((0)\begin{vmatrix} 0 & 9 \\ 0 & 10 \end{vmatrix} - (5)\begin{vmatrix} 0 & 9 \\ 0 & 10 \end{vmatrix} + (7)\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\right) \\
&\quad -(4)\left((0)\begin{vmatrix} 0 & 8 \\ 0 & 0 \end{vmatrix} - (5)\begin{vmatrix} 0 & 0 \\ 0 & 8 \end{vmatrix} + (6)\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\right) \\
&= (1)\left((5)(8)(10) + (6)(0) + (7)(0)\right) - (2)\left((0)(8)(9) - (6)(0) + (7)(0)\right) \\
&\quad +(3)\left((0)(0) - (5)(0) + (7)(0)\right) - (4)\left((0)(0) - (5)(0) + (6)(0)\right) \\
&= (1)(5)(8)(10) + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 \\
&= (1)(5)(8)(10)
\end{aligned}$$

Combining the theorem and lemma above we can now conclude

COROLLARY 13.10. *If* $\mathbf{A}$ *is an* $n \times n$ *matrix and* $\mathbf{A}'$ *is a row-echelon form of* $\mathbf{A}$ *obtained without row rescalings, then*

$$\det(\mathbf{A}) = (-1)^j \prod_{i=1}^{n} a'_{ii}$$

where j *is the total number of row-interchanges that occured in row-reducing* $\mathbf{A}$ *to* $\mathbf{A}'$*. Here* $\prod_{\iota=1}^{n} a'_{ii}$ *is the total product of the diagonal entries of* $\mathbf{A}'$*.*

COROLLARY 13.11. *A square matrix* $\mathbf{A}$ *is invertible if and only if* $\det(\mathbf{A}) \neq 0$.

THEOREM 13.12. *The determinant of a product of two square matrices is equal to the product of their determinants:*

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B}).$$