LECTURE 10

The Rank of a Matrix

Let $\mathbf{A}$ be an $m \times n$ matrix.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{mn} & \cdots & a_{mn} \end{bmatrix}$$

Recall that the **column space** of $\mathbf{A}$ is the subspace of $\mathbb{R}^m$ spanned by the columns of $\mathbf{A}$:

$$ColSp\left(\mathbf{A}\right) = span\left(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\right) \subset \mathbb{R}^m$$

where i^{th} column vector $\mathbf{c}_i$ is defined by

$$\left(\mathbf{c}_i\right)_j \equiv a_{ji} \quad , \quad j = 1, \ldots, m$$

Recalll also that the **row space** of $\mathbf{A}$ is the subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$:

$$RowSp\left(\mathbf{A}\right) = span\left(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\right) \subset \mathbb{R}^n$$

where the i^{th} row vector is defined by

$$\left(\mathbf{r}_i\right)_j = a_{ij} \quad , \quad j = 1, \ldots, n$$

A priori there is no particular relationship between the column space of $\mathbf{A}$ and the row space of $\mathbf{A}$; indeed, they are not even subspaces of the same space.

LEMMA 10.1. *If a matrix $\mathbf{A}'$ is row equivalent to a matrix $\mathbf{A}$ then the row space of $\mathbf{A}'$ is equal to the row space of $\mathbf{A}$.*

Proof. First we note that row operations can be built up from row operartions of the following form

$$R_{ij}\left(\lambda_1, \lambda_2\right) : \begin{cases} \mathbf{r}_i \rightarrow \mathbf{r}_i' = \lambda_1 \mathbf{r}_i + \lambda_2 \mathbf{r}_j & i = j \\ \mathbf{r}_i \rightarrow \mathbf{r}_i' = \mathbf{r}_i \quad , & i \neq j \end{cases} \quad , \quad \lambda_1 \neq 0 \tag{10.1}$$

For example, the interchange of i^{th} and j^{th} rows can be carried out as

$$\left\{ \begin{array}{c} \mathbf{r}_i \\ \mathbf{r}_j \end{array} \right\} \xrightarrow{R_{ij}(1,1)} \left\{ \begin{array}{c} \mathbf{r}_i' = \mathbf{r}_i + \mathbf{r}_j \\ \mathbf{r}_j' = \mathbf{r}_j \end{array} \right\} \xrightarrow{R_{ji}(-1,1)} \left\{ \begin{array}{c} \mathbf{r}_i'' = \mathbf{r}_i' = \mathbf{r}_i + \mathbf{r}_j \\ \mathbf{r}_j'' = -\mathbf{r}_j' + \mathbf{r}_i' = \mathbf{r}_i \end{array} \right\} \xrightarrow{R_{ij}(1,-1)} \left\{ \begin{array}{c} \mathbf{r}_i''' = \mathbf{r}_i'' + \mathbf{r}_j'' = \mathbf{r}_j \\ \mathbf{r}_j''' = \mathbf{r}_j'' = \mathbf{r}_i \end{array} \right\}$$

while the other two elementary row operations can be viewed as simply special cases of the row operation (**??**).

Now suppose $\mathbf{v}$ is a vector lying in the span of row vectors of $\mathbf{A}$. I will show that it also lies in the span of the row vectors of the matrix $\mathbf{A}'$ obtained by applying the row operation (**??**) to $\mathbf{A}$.

$$\begin{aligned} \mathbf{v} &\in span\left(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\right) \quad \Rightarrow \quad \mathbf{v} = c_1\mathbf{r}_1 + \cdots + c_i\mathbf{r}_i + \cdots + c_j\mathbf{r}_j + \cdots + c_m\mathbf{r}_m \\ &= c_1\mathbf{r}_1' + \cdots + c_i\left(\frac{1}{\lambda_1}\left(\mathbf{r}_i' - \lambda_2\mathbf{r}_j\right)\right) + \cdots + c_j\mathbf{r}_j' + \cdots + c_m\mathbf{r}_m' \\ &= c_1\mathbf{r}_1' + \cdots + \left(\frac{c_i}{\lambda_1}\right)\mathbf{r}_i' + \cdots + \left(c_j - \frac{c_i\lambda_2}{\lambda_1}\right)\mathbf{r}_j' + \cdots + c_m\mathbf{r}_m' \\ &\in span\left(\mathbf{r}_1', \ldots, \mathbf{r}_m'\right) \end{aligned}$$

Thus, the row spaces of $\mathbf{A}$ and $\mathbf{A}'$ are the same. If $\mathbf{A}'$ is row equivalent to $\mathbf{A}$, then by definition there must be a sequence of row operations that converts $\mathbf{A}$ into $\mathbf{A}'$.

$$\mathbf{A} \to \mathbf{A}^{(1)} \to \mathbf{A}^{(2)} \to \cdots \to \mathbf{A}^{(k)} = \mathbf{A}'$$

From the preceding paragraph, we know at each intermediate stage we have $RowSp\left(\mathbf{A}^{(i)}\right) = RowSp\left(\mathbf{A}^{(i+1)}\right)$ so we conclude

$$RowSp\left(\mathbf{A}'\right) = RowSp\left(\mathbf{A}\right)$$

LEMMA 10.2. *Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{A}'$ be its reduction to row echelon form. Then the non-zero rows of $\mathbf{A}'$ form a basis for the row space of $\mathbf{A}$.*

The basis idea underlying the proof of this lemma is best illustrated by an example. Suppose $\mathbf{A}$ is a 4×5 matrix that is row equivalent to the following matrix in reduced row-echelon form

$$\mathbf{A}'' = \begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly the span of the row vectors of $\mathbf{A}'$ is just the span of the first three row vectors (that is to say, the contribution of the last row to the row space of $\mathbf{A}$ is just $\mathbf{0}$). On the other hand, it's clear the only way we can satisfy

$$\mathbf{0} = c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3$$

is by taking $c_1 = c_2 = c_3 = 0$; because that's the only way to kill off the components of the total sum that come from the pivots of $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ (that is, we can't force a cancellation of terms coming from two different rows because only the pivot row will have a non-zero entry in the component corresponding a column with a pivot). Thus,

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 \quad \Rightarrow \quad c_1 = c_2 = c_3 = 0 \\ &\Rightarrow \quad \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\} \text{ is a basis for } span\left(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\right) = RowSp\left(\mathbf{A}''\right) = RowSp\left(\mathbf{A}\right) \end{aligned}$$

However, this isn't quite the statement of the lemma. For the lemma says the row vectors of a matrix in (un-reduced) echelon form should be a basis for the row space of $\mathbf{A}$. However, we can conclude this simply by noting that

$$\begin{aligned} \dim\left(RowSp(\mathbf{A})\right) &= \text{number of vectors in a basis for } RowSp\left(\mathbf{A}\right) \\ &= \text{number of non-zero rows in reduced echelon-form } \mathbf{A}'' \text{ of } \mathbf{A} \\ &= \text{number of non-zero rows in an echelon-form } \mathbf{A}' \text{ of } \mathbf{A} \end{aligned}$$

But because the row vectors of the matrix in echelon-form span $RowSp\left(\mathbf{A}\right)$, and because the number of these row vectors is the same as the dimension of $RowSp\left(\mathbf{A}\right)$, we can use Statement 3(b) of Theorem 9.6 (at the end of Lecture 9) to conclude that the row vectors of $\mathbf{A}'$ form a basis for $RowSp\left(\mathbf{A}\right)$.

LEMMA 10.3. *Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{A}'$ be its reduction to row echelon form. Then the columns of $\mathbf{A}$ corresponding to the columns of $\mathbf{A}'$ containing the pivots of $\mathbf{A}'$ form a basis for the column space of $\mathbf{A}$.*

This lemma is also best demonstrated by example. Suppose $\mathbf{A}$ is a 4×5 matrix that is row equivalent to the following matrix in reduced row-echelon form

$$\mathbf{A}'' = \begin{bmatrix} \mathbf{1} & 1 & 0 & 0 & 3 \\ 0 & 0 & \mathbf{1} & 0 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the pivots have been designated by bold-face type. Now the column space of $\mathbf{A}''$ will be identical to the row space of its transpose

$$(\mathbf{A}'')^T = \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 3 & 1 & 1 & 0 \end{bmatrix} \xrightarrow[R_5 \to R_5 - 3R_1 - R_3 - R_4]{R_2 \to R_2 - R_1} \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where we've used the rows containing the old pivots to clear out the rest of the matrix entries. Obviously, the remaining non-zero rows will be a basis for the $RowSp\left[(\mathbf{A}'')^T\right] = ColSp\left[\mathbf{A}''\right] = ColSp\left[\mathbf{A}\right]$. But the non-zero rows of $(\mathbf{A}'')^T$ are just (actually, linear combinations of) of the columns of $\mathbf{A}'$ containing pivots. Therefore, the columns of $\mathbf{A}'$ that contain pivots correspond to a basis for the column space of $\mathbf{A}$.

Example 10.4. Find a basis for the column space of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

- First we row reduce $\mathbf{A}$ to row-echelon form

$$\xrightarrow[\substack{R_3 \to R_3 + R_2 \\ R_4 \to R_4 + R_3}]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 2R_3]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is a row-echelon form of $\mathbf{A}$. It has pivots in the 1^{st}, 2^{nd}, and 3^{rd} columns. Therefore, the 1^{st}, 2^{nd}, and 3^{rd} columns of the original matrix $\mathbf{A}$ will form a basis for the column space of $\mathbf{A}$:

$$ColSp\,(\mathbf{A}) = span\left(\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix} \right)$$

Theorem 10.5. *Let $\mathbf{A}$ be an $m \times n$ matrix. The dimension of the row space of $\mathbf{A}$ is equal to the dimension of its column space.*

This follows easily from the preceding two lemmas since the number of non-zero rows in a matrix in row-echelon form is exactly equal to the number of columns containing pivots. This theorem leads to the following definition.

Definition 10.6. *The **rank** of a matrix is the dimension of its row space (which equals the dimension of its column space).*

Recall that the null space an $m \times n$ matrix $\mathbf{A}$ is the subspace of $\mathbb{R}^n$ corresponding to the solution space of $\mathbf{Ax} = \mathbf{0}$.

Theorem 10.7. *Let $\mathbf{A}$ be an $m \times n$ matrix. Then*

$$n = [\textit{number of columns of } \mathbf{A}] = \dim\left[\textit{Null space of } \mathbf{A}\right] + rank\,(\mathbf{A})$$

To see why this theorem must be true, consider the following simple example.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is already in reduced row-echelon form. It has three pivots so

$$rank(\mathbf{A}) = \dim\left(RowSp(\mathbf{A})\right) = \dim\left(ColSp(\mathbf{A})\right) = 3$$

The dimension of its null space is evidently 1 since the solution of the corresponding homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ implies

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \left\{ \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array} \right.$$

but leaves x_4 undetermined. Hence, the dimenion of the null space of $\mathbf{A}$ is 1. Thus,

$$4 = \text{ number of columns of } \mathbf{A} = 3 + 1 = (\text{rank of } \mathbf{A}) + \left(\dim\left(\text{null space of } \mathbf{A}\right)\right)$$

In the next lecture we shall develop a geometric interpretation of this fundamental fact.