LECTURE 9

Linear Independence and Dimension

A subspace W (for example, the solution set of a set of homogeneous linear equations) can be generated by taking linear combinations of a set of vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$. The purpose of this lecture is address the question: given a fixed subspace W, how do we know when we've picked enough vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k \in W$ so that we can represent **every** other vector in W uniquely in terms of a particular linear combination of the $\mathbf{w}_i$? In the language of Lecture 7, how do we know we have a **basis** for W?

1. Constructing a Basis for a Span of Vectors

Let $\mathbf{w}_1, \ldots, \mathbf{w}_k$ be vectors in $\mathbb{R}^n$, and let

$$W \equiv \mathit{span}\,(\mathbf{w}_1, \ldots, \mathbf{w}_k) \equiv \{\mathbf{w} \in \mathbb{R}^n \mid \mathbf{w} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k \quad ; \quad c_1, \ldots, c_k \in \mathbb{R}\} \tag{9.1}$$

Suppose $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is not a basis for W, then by Theorem 7.14 (Lecture 7), we know that we must have a non-trivial solution of

$$\mathbf{0} = r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_k\mathbf{w}_k \tag{9.2}$$

that is, a solution for which at least one of the r_i does not equal zero. Without loss of generality (e.g. by reordering the vectors $\mathbf{w}_i$) we can assume it is the last coefficient r_k that does not vanish. Then we can use (9.2) to express $\mathbf{w}_k$ in terms of the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}$

$$\mathbf{w}_k = -\frac{1}{r_k}\left(r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_{k-1}\mathbf{w}_{k-1}\right)$$

It is then easy to see that the smaller set of vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}\}$ also generate W: for $\mathbf{w} \in W$ implies

$$\begin{aligned}
\mathbf{w} &= c_1\mathbf{w}_1 + \cdots + c_{k-1}\mathbf{w}_{k-1} + c_k\mathbf{w}_k \\
&= c_1\mathbf{w}_1 + \cdots + c_{k-1}\mathbf{w}_{k-1} - \frac{c_k}{r_k}\left(r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_{k-1}\mathbf{w}_{k-1}\right) \\
&= \left(c_1 - \frac{c_k r_1}{r_k}\right)\mathbf{w}_1 + \left(c_2 - \frac{c_k r_2}{r_k}\right)\mathbf{w}_2 + \cdots + \left(c_{k-1} - \frac{c_k r_{k-1}}{r_k}\right)\mathbf{w}_{k-1} \\
&\in \mathit{span}\,(\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{k-1})
\end{aligned}$$

In other words, if $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is not a basis, we can always find a smaller subset of vectors that generate same subspace. The converse to this statement is also true: if we can not find a smaller (i.e., proper) subset of vectors that generate the subspace $W = \mathit{span}\,(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, then the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ form a basis for W.

EXAMPLE 9.1. Find a basis for $W = \mathit{span}\,(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \subset \mathbb{R}^2$ where

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad , \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad \mathbf{w}_3 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

- First we look for nontrivial solutions of

$$r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + r_3\mathbf{w}_3 = r_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + r_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + r_3\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0} \tag{9.3}$$

This vector equation is equivalent to the following linear system

$$\begin{aligned} r_1 + r_2 - 2r_3 &= 0 \\ 2r_1 + r_2 - r_3 &= 0 \end{aligned}$$

or the following augmented matrices

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \xrightarrow[R_2 \to -R_2]{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

or

$$\left.\begin{array}{r} r_1 + r_3 = 0 \\ r_2 - 3r_3 = 0 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} r_1 = -r_3 \\ r_2 = 3r_3 \end{array}\right. \qquad \text{for some } r_3 \in \mathbb{R}$$

Taking $r_3 = 1$ we thus have a solution with $r_1 = -1$ and $r_2 = 3$. Indeed,

$$(-1)\left[\begin{array}{c} 1 \\ 2 \end{array}\right] + 3\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + \left[\begin{array}{c} -2 \\ -1 \end{array}\right] = \left[\begin{array}{c} -1+3=2 \\ -2+3-1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

So

$$-\mathbf{w}_1 + 3\mathbf{w}_2 + \mathbf{w}_3 = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{w}_3 = \mathbf{w}_1 - 3\mathbf{w}_2$$

Because we can express $\mathbf{w}_3$ as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$

$$W \equiv \mathit{span}\,(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = \mathit{span}\,(\mathbf{w}_1, \mathbf{w}_2)$$

and perhaps $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for W.

To see if $\{\mathbf{w}_1, \mathbf{w}_2\}$ is indeed a basis, we repeat the calculation above. We first look for non-trivial solutions of

$$r_1\mathbf{w}_1 + r_2\mathbf{w}_2 = \mathbf{0} \tag{9.4}$$

or

$$\begin{aligned} r_1 + r_2 &= 0 \\ 2r_1 + r_2 &= 0 \end{aligned}$$

The corresponding augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow[R_2 \to -R_2]{R_1 \to R_1 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

which corresponds to a linear system with only one solution

$$\begin{aligned} r_1 &= 0 \\ r_2 &= 0 \end{aligned}$$

Since we can't find non-trivial solutions of (9.4), we conclude that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for $\mathit{span}\,(\mathbf{w}_1, \mathbf{w}_2) = \mathit{span}\,(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) \equiv W$.

The following definition formalizes the ideas behind this construction of bases.

DEFINITION 9.2. *Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a set of vectors in $\mathbb{R}^n$. A* ***dependence relation*** *among the $\mathbf{w}_i$ is an equation of the form*

$$r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_k\mathbf{w}_k = \mathbf{0} \quad , \quad \text{with at least one } r_i \neq 0.$$

If such a dependence relation exists, the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a ***linearly dependent set of vectors.*** *If such a dependence relation does not exist,then the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are said to be* ***linearly independent.***

2. Dimensions of Subspaces

THEOREM 9.3. *Let W be a subspace spanned by a set of vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ be a set of linear independent vectors in W. Then $s \leq k$.*

This theorem is kinda tedious to prove; but the basic idea behind the proof is very simple. Since vectors $\mathbf{v}_i$ lie in the span of the vectors $\mathbf{w}_j$, each of the vectors $\mathbf{v}_i$ can be written in the form

$$\mathbf{v}_i = c_{i1}\mathbf{w}_1 + c_{i2}\mathbf{w}_2 + \cdots + c_{ik}\mathbf{w}_k$$

The hypothesis that the $\mathbf{v}_i$ are linearly independent requires that the linear system

$$\begin{aligned} \mathbf{0} &= r_1\mathbf{v}_1 + \cdots + r_s\mathbf{v}_s \\ &= r_1\left(c_{11}\mathbf{w}_1 + c_{12}\mathbf{w}_2 + \cdots + c_{1k}\mathbf{w}_k\right) + \cdots + r_s\left(c_{s1}\mathbf{w}_1 + c_{s2}\mathbf{w}_2 + \cdots + c_{sk}\mathbf{w}_k\right) \\ &= \left(r_1c_{11} + \cdots + r_sc_{s1}\right)\mathbf{w}_1 + \cdots + \left(r_1c_{1k} + \cdots + r_sc_{sk}\right)\mathbf{w}_k \end{aligned}$$

Analysis of the latter system shows that it will have fewer equations (at least k) than unknowns r_s unless $s \leq k$. Hence, we can only obtain a unique solution when $s \leq k$. Hence, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ can be linearly independent only if $s \leq k$.

COROLLARY 9.4. *Any two bases of a subspace W of $\mathbb{R}^n$ have the same number of vectors.*

Proof. Suppose that a set B with k vectors and a set B' with k' vectors were both bases for a subspace W. Then both B and B' are linearly independent sets of vectors, and the vectors in either set span W. Regarding B as a set of vectors spanning W and B' as a set of linearly independent vectors in W, the preceding theorem tells us that $k' \leq k$. On the other hand, regarding B' as a set of vectors spanning W and B as a set of linearly independent vectors in W, the preceding theorem tells us that $k \leq k'$. We conclude that $k = k'$.

DEFINITION 9.5. *Let W be a subspace of $\mathbb{R}^n$. The number of elements in any basis for W is the* ***dimension*** *of W.*

THEOREM 9.6. *Existence and Determination of Bases*

1. *Every subspace of W of $\mathbb{R}^n$ has a basis and $\dim(W) \leq n$.*
2. *Every linearly independent set of vectors in $\mathbb{R}^n$ can be enlarged, if necessary, to become a basis for $\mathbb{R}^n$.*
3. *If W is a subspace of $\mathbb{R}^n$ and $\dim(W) = k$, then*
   (a) *every linearly independent set of k vectors in W is a basis for W.*
   (b) *every set of k vectors in W that spans W is a basis for W.*