LECTURE 7

Subspaces of $\mathbb{R}^n$

1. Subspaces

DEFINITION 7.1. *A subset W of $\mathbb{R}^n$ is said to be* ***closed under vector addition*** *if for all $\mathbf{u}, \mathbf{v} \in W$, $\mathbf{u}+\mathbf{v}$ is also in W. If $r\mathbf{v}$ is in W for all vectors $\mathbf{v} \in W$ and all scalars $r \in \mathbb{R}$, then we say that W is closed under scalar multiplication. A non-empty subset W of $\mathbb{R}^n$ that is closed under both vector addition and scalar multiplication is called a* ***subspace*** *of $\mathbb{R}^n$.*

EXAMPLE 7.2. Let $\mathbf{u} = (1,0)$ and $\mathbf{v} = (0,2)$ be vectors in $\mathbb{R}^2$. We can construct a subset that closed under vector addition as follows.

$$W_0 = \{\mathbf{w} \in \mathbb{R}^n \mid \mathbf{w} = j\mathbf{u} + k\mathbf{v} \quad ; \quad j,k \text{ positive integers}\}$$

To see that this set is closed under vector addition, let $\mathbf{w}, \mathbf{w}' \in W_0$. Then

$$\begin{aligned} \mathbf{w} &= j\mathbf{u} + k\mathbf{v} \\ \mathbf{w}' &= j'\mathbf{u} + k'\mathbf{v} \end{aligned}$$

for some positive integers j, k, j', and k'. But then there are positive integers j, k, j' and k' such that

$$\mathbf{w} + \mathbf{w}' = (j\mathbf{u} + k\mathbf{v}) + (j'\mathbf{u} + k'\mathbf{v}) = (j + j')\,\mathbf{u} + (k + k')\,\mathbf{v} \in \mathbf{W}$$

because both $(j + j')$ and $(k + k')$ ar e positive integers if j, k, j', and k' are positive integers.

The set W_0 is not a subspace, however; because it is not closed under scalar multiplication. To see this, note that the vector

$$\frac{1}{2}\mathbf{u} = \left(\frac{1}{2}, 0\right)$$

can not be represented as sum of $\mathbf{u}$ and $\mathbf{v}$ with positive integer coefficients.

EXAMPLE 7.3. The preceding example, however, does provide a clue as to one way to constructing a subspace. Let $\mathbf{u} = (1,0)$ and $\mathbf{v} = (0,2)$ be vectors in $\mathbb{R}^2$. Consider the set

$$W = \{\mathbf{w} \in \mathbb{R}^n \mid \mathbf{w} = j\mathbf{u} + k\mathbf{v} \quad ; \quad j,k \in \mathbb{R}\}$$

This is closed under vector addition because if $\mathbf{w}, \mathbf{w}' \in W$, then there are real numbers r, s, r' and s' such that

$$\begin{aligned} \mathbf{w} &= r\mathbf{u} + s\mathbf{v} \\ \mathbf{w}' &= r'\mathbf{u} + s'\mathbf{v} \end{aligned}$$

But then

$$\mathbf{w} + \mathbf{w}' = (r + r')\,\mathbf{u} + (s + s')\,\mathbf{v} \in W$$

since $(r + r') \in \mathbb{R}$ and $(s + s') \in \mathbb{R}$. And, for any real number t

$$t\mathbf{w} = (tr)\mathbf{u} + (ts)\mathbf{v} \in W$$

since $(tr) \in \mathbb{R}$ and $(ts) \in \mathbb{R}$.

The following theorem generalizes this last example.

THEOREM 7.4. *Let* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *be a set of vectors in* $\mathbb{R}^n$. *Then the span of* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is a subspace of* $\mathbb{R}^n$.

Proof. Recall that the span of a set of vector is the set of all possible linear combinations of those vectors. Set

$$W = span\left(\mathbf{v}_1, \ldots, \mathbf{v}_k\right) \equiv \{\mathbf{w} \in \mathbb{R}^n \mid \mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \quad ; \quad c_1, c_2, \ldots, c_k \in \mathbb{R}\}$$

Then for any vectors $\mathbf{w}, \mathbf{w}' \in W$ we have

$$\begin{aligned} \mathbf{w} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \\ \mathbf{w}' &= c_1'\mathbf{v}_1 + c_2'\mathbf{v}_2 + \cdots + c_k'\mathbf{v}_k \end{aligned}$$

for some choice of real numbers $c_1, \ldots, c_k$ and $c_1', \ldots, c_k'$. But then

$$\mathbf{w} + \mathbf{w}' = (c_1 + c_1')\,\mathbf{v}_1 + (c_2 + c_2')\,\mathbf{v}_2 + \cdots + (c_k + c_k')\,\mathbf{v}_k \in W$$

and if t is any real number

$$t\mathbf{w} = (tc_1)\mathbf{v}_1 + (tc_2)\,\mathbf{v}_2 + \cdots + (tc_k)\,\mathbf{v}_k \in W$$

REMARK 7.5. We shall often refer to the span of a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors in $\mathbb{R}^n$ as the **subspace generated by** $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

2. Solutions of Homogeneous Systems

We now come to another fundamental way of realizing a subspace of $\mathbb{R}^n$.

DEFINITION 7.6. *A linear system of the form* $\mathbf{Ax} = \mathbf{0}$ *is called* ***homogeneous***.

A homogeneous linear system is always solvable since $\mathbf{x} = \mathbf{0}$ is always a solution. As such, this solution is not very interesting; we call it the **trivial solution**. A homogeneous linear system may possess other **non-trivial** solutions (i.e. solutions $\mathbf{x} \neq \mathbf{0}$), this is where we shall focus our attention today.

LEMMA 7.7. *Suppose* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are solutions of a homogeneous system* $\mathbf{Ax} = \mathbf{0}$. *Then so is any linear combination* $r\mathbf{x}_1 + s\mathbf{x}_2$ *of* $\mathbf{x}_1$ *and* $\mathbf{x}_2$.

Proof. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are solution of $\mathbf{Ax} = \mathbf{0}$ we have

$$\mathbf{Ax}_1 = \mathbf{0} = \mathbf{Ax}_2$$

But then

$$\begin{aligned} \mathbf{A}\left(r\mathbf{x}_1 + s\mathbf{x}_2\right) &= \mathbf{A}(r\mathbf{x}_1) + \mathbf{A}(s\mathbf{x}_2) \\ &= r\left(\mathbf{Ax}_1\right) + s\left(\mathbf{Ax}_2\right) \\ &= r\mathbf{0} + s\mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

so $r\mathbf{x}_1 + s\mathbf{x}_2$ is also a solution.

THEOREM 7.8. *The solution space of a homogeneous linear system is a subspace of* $\mathbb{R}^n$.

Proof. The preceding lemma demonstrates that the solution space of a homogeneous linear system is closed under both vector addition (take $r = 1$ and $s = 1$ in the proof of the preceding lemma) and scalar multiplication (let r be any real number and take $s = 0$, in the proof of the lemma) . Therefore, it is a subspace of $\mathbb{R}^n$.

3. Subspaces Associated with Matrices

DEFINITION 7.9. *The* ***row space*** *of an* $m \times n$ *matrix* $\mathbf{A}$ *is the span of row vectors of* $\mathbf{A}$.

Since the row vectors of an $m \times n$ matrix are n-dimensional vectors, the row space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

DEFINITION 7.10. *The* ***column space*** *of an* $m \times n$ *matrix* $\mathbf{A}$ *is the span of column vectors of* $\mathbf{A}$.

Since the column vectors of an $m \times n$ matrix are m-dimensional vectors, the column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

DEFINITION 7.11. *The* ***null space*** *of an* $m \times n$ *matrix* $\mathbf{A}$ *is the solution set of homogeneous linear system* $\mathbf{Ax} = \mathbf{0}$.

By the theorem of the proceding section, the null space of an $m \times n$ matrix $\mathbf{A}$ will be a subspace of $\mathbb{R}^n$.

Consider now a non-homogeneous linear system

$$\mathbf{Ax} = \mathbf{b}$$

The left hand side of such an equation is

$$\begin{aligned}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} &= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} \\
&= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}
\end{aligned}$$

The final expression on the right hand side is evidently a linear combination of the column vectors of $\mathbf{A}$. The consistency of the equation $\mathbf{Ax} = \mathbf{b}$ then requires the column vector $\mathbf{b}$ to also lie within the span of the column vectors of $\mathbf{A}$. Thus we have

THEOREM 7.12. *A linear system* $\mathbf{Ax} = \mathbf{b}$ *is consistent if and only if* $\mathbf{b}$ *lies in the column space of* $\mathbf{A}$.

4. Bases

Consider the subspace generated by the following three vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad , \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad , \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

It turns out that this is the same as the subspace generated from just $\mathbf{v}_1$ and $\mathbf{v}_2$. To see this note that

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \mathbf{v}_1 + (-1)\mathbf{v}_2$$

But any vector $\mathbf{w}$ in $span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is expressible in the form

$$\begin{aligned} \mathbf{w} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \\ &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(\mathbf{v}_1 - \mathbf{v}_2) \\ &= (c_1 + c_3)\mathbf{v}_1 + (c_2 - c_3)\mathbf{v}_2 \\ &\in span(\mathbf{v}_1, \mathbf{v}_2) \end{aligned}$$

For reasons of effficiency alone, it is natural to try to find the minimum number of vectors needed to specify every vector in a subspace W. Such a set will be called a **basis** for W.

There is also another reason to be interested in basis vectors. Consider the vector

$$\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

In terms of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, we can write this as either

$$\mathbf{w} = 3\mathbf{v}_1 - \mathbf{v}_2 - 2\mathbf{v}_3$$

or

$$\mathbf{w} = -2\mathbf{v}_1 + 4\mathbf{v}_2 + 3\mathbf{v}_3$$

or even

$$\mathbf{w} = -2\mathbf{v}_1 + 4\mathbf{v}_2 + 3\mathbf{v}_3$$

However, there is only one way to represent $\mathbf{w}$ as a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. For the condition $\mathbf{w} = c_1\mathbf{v}_1 - c_2\mathbf{v}_2$ requires

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_2 \\ c_1 \end{bmatrix}$$

is equivalent to the following linear system

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_2 &= 1 \\ c_1 &= 1 \end{aligned}$$

which obviously as $c_1 = 1$ and $c_2 = 1$ as its only solution.

This motivates the following definition.

DEFINITION 7.13. *Let W be a subspace of $\mathbb{R}^n$. A subset $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k\}$ of W is called a **basis** for W if every vector in W can be uniquely expressed as linear combination of the vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$.*

THEOREM 7.14. *A set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k\}$ is a basis for the subspace W generated by $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k\}$ if and only if*

$$r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_k\mathbf{w}_k = \mathbf{0} \qquad \textit{implies} \quad 0 = r_1 = r_2 = \cdots = r_k$$

Proof.

$\Rightarrow$ Suppose $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k\}$ is a basis for $W = span(\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k)$. Then every vector in W can be uniquely specfied as a vector of the form

$$\mathbf{v} = r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_k\mathbf{w}_k \quad ; \quad r_1, r_2, \ldots, r_k \in \mathbb{R}$$

In particular, the zero vector

$$\mathbf{0} = (0)\mathbf{w}_1 + (0)\mathbf{w}_2 + \cdots + (0)\mathbf{w}_k \tag{7.1}$$

lies in W. Because $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k\}$ is assumed to be a basis, the linear combination on the right hand side of (7.1) must be the unique linear combination of the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ that is equal to $\mathbf{0}$. Hence,

$$r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_k\mathbf{w}_k = \mathbf{0} \qquad \text{implies} \quad 0 = r_1 = r_2 = \cdots = r_k$$

$\Leftarrow$ Suppose

$$r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_k\mathbf{w}_k = \mathbf{0} \qquad \text{implies} \quad 0 = r_1 = r_2 = \cdots = r_k$$

We want to show that $\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k\}$ is a basis for $W = span\,(\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k)$. In other words, we need to show that there is only one choice of coefficients $r_1, \ldots, r_k$ such that a vector $\mathbf{v} \in W$ can be expressed in the form $\mathbf{v} = r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_k\mathbf{w}_k$. Suppose there were in fact two distinct ways of representing $\mathbf{v}$:

$$\mathbf{v} = r_1\mathbf{w}_1 + r_2\mathbf{w}_2 + \cdots + r_k\mathbf{w}_k \tag{7.2}$$

$$\mathbf{v} = s_1\mathbf{w}_1 + s_2\mathbf{w}_2 + \cdots + s_k\mathbf{w}_k \tag{7.3}$$

Subtracting the second equation from the first yields

$$\mathbf{0} = (r_1 - s_1)\,\mathbf{w}_1 + (r_2 - s_2)\,\mathbf{w}_2 + \cdots + (r_k - s_k)\,\mathbf{w}_k$$

Our hypothesis now implies

$$0 = r_1 - s_1 = r_2 - s_2 = \cdots = r_k - s_k$$

In other words

$$\begin{aligned} r_1 &= s_1 \\ r_2 &= s_2 \\ &\vdots \\ r_k &= s_k \end{aligned}$$

and so the two linear combinations on the right hand sides of (7.2) and (7.3) must be identical.

THEOREM 7.15. *Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following statements are equivalent.*

1. *The linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for each vector $\mathbf{b} \in \mathbb{R}^n$.*
2. *The matrix $\mathbf{A}$ is row equivalent to the identity matrix.*
3. *The matrix $\mathbf{A}$ is invertible.*
4. *The column vectors of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.*

Proof. We have already demonstrated the equivalence of statements 2, 3 and 4. It therefore suffices to show that statement 4 is equivalent to statement 1.

To see that statement 4 implies statement 1, suppose that the column vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ of $\mathbf{A}$ form a basis for $\mathbb{R}^n$. Then by Theorem 7.12, the linear system $\mathbf{Ax} = \mathbf{b}$ is consistent for all vectors $\mathbf{b} \in span\,(\mathbf{c}_1, \ldots, \mathbf{c}_n) = \mathbb{R}^n$. But a direct calculation reveals

$$\mathbf{b} = \mathbf{Ax} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n$$

Because the vectors $\mathbf{c}_1, \ldots, \mathbf{c}_n$ form a basis, there choice of coefficients $x_1, x_2, \ldots, x_n$ must be unique. Therefore, the linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for each vector $\mathbf{b} \in \mathbb{R}^n$.

On the other hand, suppose the linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution for each vector $\mathbf{b} \in \mathbb{R}^n$. In particular, this must be true for $\mathbf{b} = \mathbf{0}$. Therefore, there is only one choice of coefficients $x_1, x_2, \ldots, x_n$ such that

$$\mathbf{0} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{Ax}$$

By the preceding theorem we can conclude that the column vectors of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.

EXAMPLE 7.16. Show that the vectors $\mathbf{v}_1 = (1, 1, 3)$, $\mathbf{v}_2 = (3, 0, 4)$, and $\mathbf{v}_3 = (1, 4, -1)$ form a basis for $\mathbb{R}^3$.

By the preceding theorem, it suffices to show that the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 4 \\ 3 & 4 & -1 \end{bmatrix}$$

is invertible. Row reducing $\mathbf{A}$ yields

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 4 \\ 3 & 4 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -3 & 3 \\ 0 & -5 & -4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{5}{3}R_2} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & -9 \end{bmatrix}$$

The matrix on the far right is upper triangular so it's obviously invertible. Hence, $\mathbf{A}$ is invertible; hence the column vectors of $\mathbf{A}$ form a basis for $\mathbb{R}^3$; hence the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

The preceding theorem is applicable only to square matrices $\mathbf{A}$ and linear systems of n equations in n unknowns. It can be extended to more general matrices and linear systems in the following manner.

THEOREM 7.17. *Let $\mathbf{A}$ be an $m \times n$ matrix. Then the following are equivalent.*

1. *Each consistent system $\mathbf{Ax} = \mathbf{b}$ has a unique solution.*
2. *The reduced row echelon form of $\mathbf{A}$ consists of the $n \times n$ identity matrix followed by $m - n$ rows containing only zeros.*
3. *The column vectors of $\mathbf{A}$ form a basis for the column space of $\mathbf{A}$.*

Proof.

$1 \iff 2$: From Theorem 5.8 of Lecture 5 (Theorem 1.7 in text), we know that a consistent linear system $\mathbf{Ax} = \mathbf{b}$ has a unique solution if and only if $\mathbf{A}$ is row equivalent to a matrix $\mathbf{A}'$ in row-echelon form such that every column of $\mathbf{A}'$ has a pivot. Since $\mathbf{A}$, and hence $\mathbf{A}'$, has n columns, we can conclude that the solution of every consistent linear system $\mathbf{Ax} = \mathbf{b}$ is unique if and only if we have must have n pivots. In order to have n pivots the number m of rows must be $\geq n$. When $n = m$ there will be one pivot for each row, and the pivots will all reside along the diagonal, like so

$$\mathbf{A}' = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

occuring in at least n rows. If $\mathbf{A}'$ is further reduced to a matrix $\mathbf{A}''$ in **reduced** row-echelon form, then all the pivots are re-scaled to 1 and all the entries above the pivots are equal to 0. Thus,

$$\mathbf{A}'' = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \mathbf{I}$$

If $m > n$, we still require n pivots, that means the only way we can consistently add rows to the picture above is by adding rows without pivots; i.e., rows containing only 0's.

$1 \iff 3$: Suppose $\mathbf{Ax} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$ in the column space $C(\mathbf{A})$ of $\mathbf{A}$. (Recall $\mathbf{b}$ must lie in the column space of $\mathbf{A}$ in order for the linear system to be consistent.) Then, if we denote the column vectors of $\mathbf{A}$ by $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ we have

$$\mathbf{b} = \mathbf{Ax} \equiv x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n \quad , \quad \text{for all } \mathbf{b} \in C(\mathbf{A}) \equiv span\,(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n)$$

If the solution $\mathbf{x}$ is unique, then there is only one such linear combination of the column vectors $\mathbf{c}_i$ for each vector $\mathbf{b} \in C(\mathbf{A})$. Hence, the column vectors $\mathbf{c}_i$ provide a basis for $C(\mathbf{A})$. On the other hand, if the column vectors were not a basis for $C(\mathbf{A}) \equiv span\,(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n)$, then that would mean that there are vectors $\mathbf{b}$

lying in $C(\mathbf{A})$ such that the expansion $\mathbf{b} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n$ in terms of the $\mathbf{c}_i$ is not unique. Hence, a solution $\mathbf{x}$ to $\mathbf{Ax} = \mathbf{b}$ would not be unique.

THEOREM 7.18. *Let $\mathbf{Ax} = \mathbf{b}$ be a non-homogeneous linear system, and let $\mathbf{p}$ be any particular solution of this sytem. Then every solution of $\mathbf{Ax} = \mathbf{b}$ can be expressed in the form*

$$\mathbf{x} = \mathbf{p} + \mathbf{h}$$

where $\mathbf{h}$ is a solution of the corresponding homogeneous system $\mathbf{Ax} = \mathbf{0}$.

Proof. Suppose $\mathbf{p}$ and $\mathbf{x}_1$ are both solutions of $\mathbf{Ax} = \mathbf{b}$. Then set

$$\mathbf{h} = \mathbf{x}_1 - \mathbf{p}$$

Then $\mathbf{h}$ satisfies

$$\mathbf{Ah} = \mathbf{A}(\mathbf{x}_1 - \mathbf{p}) = \mathbf{Ax}_1 - \mathbf{Ap} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Hence, $\mathbf{x}_1 = \mathbf{p} + \mathbf{h}$ with $\mathbf{h}$ a solution of $\mathbf{Ax} = \mathbf{0}$.