LECTURE 6

Inverses of Square Matrices

1. Introduction

To motivate our discussion of matrix inverses, let me recall the solution of a linear equation in one variable:

$$ax = b \tag{6.1}$$

This is achieved simply by multiplying both sides by a^{-1}. Put another way, in more formal language, to solve (6.1) we multiply both sides by the multiplicative inverse of a.

In the preceding lectures we have seen that, by adopting a matrix formulation, we can rewrite a linear system consisting of m equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{6.2}$$

as a matrix equation

$$\mathbf{Ax} = \mathbf{b} \tag{6.3}$$

which, at least notationally, has the same form as (6.1). In fact, as a result of our fore-sighted choice of matrix notation, we can actually solve (6.3) in the same manner as we solved (6.1) whenever we can find a multiplicative inverse $\mathbf{A}^{-1}$ of the matrix.

However, before we try to push this analogy too far, let me point out its limitations. In the case of real numbers, every number except 0 has a multiplicative inverse; however, it is not true that every non-zero matrix has an inverse. In fact, in general matrices **do not** have inverses. For if $\mathbf{A}$ is an $m \times n$ matrix then we cannot have a $r \times s$ matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$$

unless $m = n = r = s$ (otherwise, one of the products $\mathbf{A}^{-1}\mathbf{A}$ or $\mathbf{A}\mathbf{A}^{-1}$ is not defined). And even when we restrict attention to square matrices (i.e. $n \times n$ matrices), we can find non-zero matrices that do not have inverses. For example, to find an inverse of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

we would need to find a matrix

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix}$$

Looking at the entries in the first column of the first row on both sides we see that this requires in particular that $1 = 0$; an obvious contradiction.

2. Properties of Matrix Inverses

Before we actually learn how to compute matrix inverses, I shall outline some of their elementary properties.

DEFINITION 6.1. *An $n \times n$ matrix $\mathbf{A}$ is **invertible** if there exists an $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$, the $n \times n$ identity matrix. Such a matrix $\mathbf{C}$ is called an **inverse** of $\mathbf{A}$. If an $n \times n$ matrix $\mathbf{A}$ is not invertible, it is called **singular**.*

THEOREM 6.2. *If an $n \times n$ matrix is invertible, then its inverse is unique.*

Proof. Let $\mathbf{C}$ and $\mathbf{D}$ be matrices such that $\mathbf{AC} = \mathbf{I}$ and $\mathbf{DA} = \mathbf{I}$. Then, one the one hand, we have

$$\mathbf{D}(\mathbf{AC}) = (\mathbf{DA})\,\mathbf{C} = (\mathbf{I})\mathbf{C} = \mathbf{C}$$

and, on the other,

$$\mathbf{D}(\mathbf{AC}) = \mathbf{D}(\mathbf{I}) = \mathbf{D}$$

and so

$$\mathbf{C} = \mathbf{D}$$

NOTATION 6.3. Henceforth we shall denote the unique inverse of an $n \times n$ matrix $\mathbf{A}$ by $\mathbf{A}^{-1}$.

THEOREM 6.4. *Let $\mathbf{A}$ and $\mathbf{B}$ be invertible $n \times n$ matrices. Then their product $\mathbf{AB}$ is also invertible and*

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Proof. A direct computation shows that

$$\begin{aligned}
\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)(\mathbf{AB}) &= \mathbf{B}^{-1}\left(\mathbf{A}^{-1}\mathbf{A}\right)\mathbf{B} \\
&= \mathbf{B}^{-1}(\mathbf{I})\,\mathbf{B} \\
&= \mathbf{B}^{-1}(\mathbf{IB}) \\
&= \mathbf{B}^{-1}\mathbf{B} \\
&= \mathbf{I}
\end{aligned}$$

and similarly

$$(\mathbf{AB})\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) = \mathbf{I}$$

Since matrix inverses are unique, we can conclude that $\mathbf{AB}$ is invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

LEMMA 6.5. *Every elementary matrix has an inverse.*

Proof. Recall that an elementary matrix is a matrix that is obtained from an identity matrix by a single elementary row operation.

Let $\mathbf{E}$ be the elementary matrix corresponding to the row operation that exchanges the i^{th} and j^{th} rows. Then if exchange the the i^{th} and j^{th} rows again, we get back where we started; in other words we must have

$$\mathbf{EE} = \mathbf{I}$$

so $\mathbf{E}$ is its own inverse.

Now suppose $\mathbf{E}$ is the elementary matrix corresponding to the rescaling of a particular row by a factor $k \neq 0$, and $\mathbf{E}'$ is the elementary matrix corresponding to rescaling that same row by a factor k^{-1}. Then we have

$$\mathbf{EE}' = \mathbf{I}$$

So elementary matrices corresponding to rescalings have inverses as well.

Finally let $\mathbf{E}$ be the elementary matrix corresponding to replacing the j^{th} row with its sum with k times the i^{th} row. This operation can be undone by replacing the j^{th} row by its sum with $-k$ times the i^{th} row. Let $\mathbf{E}'$ be the elementary matrix corresponding to this latter row operation. Then we have

$$\mathbf{E}\mathbf{E}' = \mathbf{I}$$

so elementary matrices corresponding to the replacements of rows by their sums with multiples of other rows have inverses.

Thus, every elementary matrix has an inverse.

Remark 6.6. In fact, as the above proof shows, **the inverse of an elementary matrix is also an elementary matrix**.

Lemma 6.7. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. The linear system* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *has a solution for every choice of column vector* $\mathbf{b}$ *if and only if* $\mathbf{A}$ *is row equivalent to the identity matrix.*

Proof.

$\Leftarrow$ Suppose $\mathbf{A}$ is row equivalent to the identity matrix. This means that by a sequence of row operatations we can transform the augmented matrix

$$[\mathbf{A} \mid \mathbf{b}] \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \tag{6.4}$$

to the form

$$[\mathbf{I} \mid \mathbf{b}'] = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b'_1 \\ 0 & 1 & \cdots & 0 & b'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b'_n \end{array}\right] \tag{6.5}$$

The solution of the linear system corresponding to (6.5) is obviously

$$\mathbf{x} = \left[\begin{array}{c} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{array}\right] \tag{6.6}$$

However, since solution sets of the linear systems corresponding (6.4) and (6.5) must be the same, (6.6) must also be a solution of the linear system corresponding ot (6.4): i.e. a solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$.

$\Rightarrow$ (Proof by Contradiction). Suppose that by row operations we can reduce the augmented matrix $[\mathbf{A} \mid \mathbf{b}]$ to an augmented matrix $[\mathbf{A}' \mid \mathbf{b}']$ where $\mathbf{A}'$ is an $n \times n$ matrix in reduced row echelon form. If $\mathbf{A}'$ is not the identity matrix, then the last row of $\mathbf{A}'$ will contain only 0's. If we can demonstrate that for some choice of $\mathbf{b}$ the column vector $\mathbf{b}'$ has a non-zero entry in the last row, we will have demonstrated (by virtue of part 1 of the theorem at the end of the preceding lecture), that the corresponding linear system has no solution. Now since $\mathbf{A}'$ was obtained from $\mathbf{A}$ by row reduction, we have $\mathbf{A}' = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$ for some set of elementary matrices $\mathbf{E}_i$. We must also have $\mathbf{b}' = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{b}$. Suppose we choose $\mathbf{b}$ so that

$$\mathbf{b} = (\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1)^{-1}\mathbf{e}_n$$

where $\mathbf{e}_n$ is the column vector with a 1 in the last row and 0's everywhere else. (Note: every elementary matrix is invertible, and so is the product of any set of any set of invertible matrices. Thus, the right hand side is well-defined). We then have

$$\mathbf{b}' = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{b} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1(\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1)^{-1}\mathbf{e}_n = \mathbf{e}_n$$

Thus, we **can** find a choice of $\mathbf{b}$ such that $\mathbf{b}'$ has a non-zero entry in the last row. Hence, if $\mathbf{A}'$ is not the identity matrix, for some choices of $\mathbf{b}$ we are lead to contradictory linear systems.

The following theorem tells us that it is really only necessary to check that $\mathbf{AC} = \mathbf{I}$ to verify that one $n \times n$ matrix is the inverse of another.

THEOREM 6.8. *Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Then $\mathbf{AC} = \mathbf{I}$ if and only if $\mathbf{CA} = \mathbf{I}$.*

3. Calculation of Matrix Inverses

Let us now turn to the problem of calculating the inverse of a square matrix. Suppose we start with a matrix $\mathbf{A}$ and apply elementary row operations until we produce a matrix that is not only in reduced row-echelon form, but in fact the identity matrix. This would imply that there would be a corresponding sequence $\{\mathbf{E}_i\}$ of elementary matrices such that

$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{I}$$

In other words,

$$\mathbf{A}^{-1} = \mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1$$

However,

$$\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1 = (\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_2\mathbf{E}_1)\,\mathbf{I}$$

So we can obtain $\mathbf{A}^{-1}$ by applying the same sequence of row operations to the identity matrix $\mathbf{I}$.

This observation leads us to the following procedure.

1. Form the augmented matrix $[\mathbf{A} \mid \mathbf{I}]$.
2. Apply the Gauss-Jordan method to attempt to reduce $[\mathbf{A} \mid \mathbf{I}]$ to the form $[\mathbf{I} \mid \mathbf{C}]$.
3. If successful, then $\mathbf{C} = \mathbf{A}^{-1}$. Otherwise, $\mathbf{A}^{-1}$ does not exist.

EXAMPLE 6.9. Calculate the inverse of

$$\mathbf{A} = \left[\begin{array}{cc} 1 & 1 \\ 2 & 3 \end{array}\right]$$

We set

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]$$

We'll use the notation $R_i \to R_i + (k)R_j$ to indicate the elementary row operation corresponding to replacing the i^{th} row with its sum with k times the j^{th} row:

$$\begin{array}{rcll} R_2 & \to & R_2 + (-2)R_1 & \Rightarrow \quad \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \\ R_1 & \to & R_1 - R_2 & \Rightarrow \quad \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right] \end{array}$$

The matrix in the first block is now the 2×2 unit matrix. The matrix in the second block should then be the inverse of $\mathbf{A}$. Let's confirm this:

$$\begin{aligned} \left[\begin{array}{cc} 1 & 1 \\ 2 & 3 \end{array}\right]\left[\begin{array}{cc} 3 & -1 \\ -2 & 1 \end{array}\right] &= \left[\begin{array}{cc} (1,1)\cdot(3,-2) & (1,1)\cdot(-1,1) \\ (2,3)\cdot(3,-2) & (2,3)\cdot(-1,1) \end{array}\right] \\ &= \left[\begin{array}{cc} 3-2 & -1+1 \\ 6-6 & -2+3 \end{array}\right] \\ &= \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \\ &\equiv \mathbf{I} \end{aligned}$$

Hence,

$$\mathbf{A}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

4. Existence of Matrix Inverses

At the beginning of this lecture I gave several examples of situations where matrix inverses do not exist. We'll close today's lecture with a theorem that relates the existence of

THEOREM 6.10. *Let $\mathbf{A}$ be an $n \times n$ matrix. Then the following conditions are equivalent.*

1. $\mathbf{A}$ *is invertible.*
2. $\mathbf{A}$ *is row equivalent to the identity matrix.*
3. *The linear system $\mathbf{Ax} = \mathbf{b}$ has a solution for each n-component column vector $\mathbf{b}$.*
4. $\mathbf{A}$ *can be expressed as a product of elementary matrices.*
5. *The span of the column vectors of $\mathbf{A}$ is $\mathbb{R}^n$.*