LECTURE 4

Matrices and Matrix Algebra, Cont'd

1. Examples of Matrix Multiplication

Recall from the preceding lecture our definition of matrix multiplication.

DEFINITION 4.1. *Let* $\mathbf{A}$ *be an* m *by* n *matrix and let* $\mathbf{B}$ *be an* s *by* t *matrix. If* $n \neq s$ *the matrix product* $\mathbf{AB}$ *is not defined (i.e. if the number of columns of* $\mathbf{A}$ *does not equal the numbe of rows of* $\mathbf{B}$*, the matrix product is not defined). If* $n = s$*, then the matrix product* $\mathbf{AB}$ *is defined and is the* m *by* t *matrix whose entries* $(\mathbf{AB})_{ij}$ *are prescribed by*

$$\begin{aligned}(\mathbf{AB})_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \\ &= \sum_{k=1}^{n} a_{ik}b_{kj}\end{aligned}$$

In other words, the entry in j^{th} *column of the* i^{th} *row of the product matrix* $\mathbf{AB}$ *is the dot product the vector correspondind to the* i^{th} *row of* $\mathbf{A}$ *and the vector corresponding to the* j^{th} *column of* $\mathbf{B}$.

Let's now compute some illustrative examples

EXAMPLE 4.2.

$$\left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right]\left[\begin{array}{cc} 1 & -1 \\ 2 & -1 \\ 1 & -2 \end{array}\right] \qquad \text{does not exist}$$

Because we need the same number of columns in the first factor as there are rows in the second factor.

EXAMPLE 4.3.

$$\left[\begin{array}{cc} 1 & 2 \\ -1 & 2 \end{array}\right]\left[\begin{array}{c} 1 \\ -1 \end{array}\right] = \left[\begin{array}{c} -1 \\ -3 \end{array}\right]$$

$$\left[\begin{array}{cc} 1 & -1 \end{array}\right]\left[\begin{array}{cc} 1 & 2 \\ -1 & 2 \end{array}\right] = \left[\begin{array}{cc} 2 & 0 \end{array}\right]$$

So even though the 2 by 1 matrix $\left[\begin{array}{c} 1 \\ -1 \end{array}\right]$ and the 1 by 2 matrix $\left[\begin{array}{cc} 1 & -1 \end{array}\right]$ correspond to the same 2-dimensional vector $(1,-1)$, their products with the 2 by 2 matrix $\left[\begin{array}{cc} 1 & 2 \\ -1 & 2 \end{array}\right]$ are not the same.

EXAMPLE 4.4.

$$\left[\begin{array}{cc} 2 & 1 \\ -1 & 1 \end{array}\right]\left[\begin{array}{cc} 1 & -1 \\ -1 & 2 \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ -2 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc} 1 & -1 \\ -1 & 2 \end{array}\right]\left[\begin{array}{cc} 2 & 1 \\ -1 & 1 \end{array}\right] = \left[\begin{array}{cc} 3 & 0 \\ -4 & 1 \end{array}\right]$$

So the product **AB** of two matrices **A** and **B** is not necessarily the same as the product **BA**. In other words, matrix multiplication is not commutative in general. Indeed, it can happen that **AB** exists but **BA** is not even defined.

Note that this circumstances partially explains the paradox of the first example. Let **A** denote the 2 by 2 matrix $\left[\begin{array}{cc} 1 & 2 \\ -1 & 2 \end{array}\right]$. If we interprete the vector $(1,-1)$ as a 2 by 1 matrix $\mathbf{v}$, then only the product $\mathbf{Av}$ is defined; and if we interprete the vector $(1,-1)$ as a 1 by 2 matrix then only the product $\mathbf{vA}$ is defined

EXAMPLE 4.5.

$$\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right]\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right]=\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$$

Recall that for real numbers $x^2=0$ implies $x=0$. This is evidently not the case for matrices: it can happen that $\mathbf{A}^2=\mathbf{0}$ but **A** is not equal to the zero matrix **0**.

EXAMPLE 4.6.

$$\left[\begin{array}{cc} -1 & 1 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]=\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$$

Recall that for real numbers $xy=0$ implies either $x=0$ or $y=0$. This is evidently not the case for matrices: it can happen that $\mathbf{AB}=\mathbf{0}$ but neither **A** or **B** is equal to the zero matrix **0**.

EXAMPLE 4.7.

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right]=\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right]$$

$$\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right]\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]=\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right]$$

And so muliplying any 3 by 3 matrix **A** by the matrix

$$\mathbf{I}=\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

just replicates the matrix **A**:

$$\mathbf{AI}=\mathbf{IA}=\mathbf{A}$$

The example above generalizes to arbitrary n by n matrices (i.e. "square matrices"). This motivates the following definition.

DEFINITION 4.8. *Let* **I** *be the* n *by* n *matrix whose entries are given by*

$$I_{ij}=\left\{\begin{array}{ll} 1 & \textit{if } i=j \\ 0 & \textit{if } i\neq j \end{array}\right.$$

In other words, **I** *is an* n *by* n *matrix with* 1*'s along the diagonal (running from the upper left to the lower right) and* 0*'s everywhere else. We call such a matrix the* n *by* n ***identity matrix****. It has the property that* $\mathbf{IA}=\mathbf{AI}=\mathbf{A}$ *for all* n *by* n *matrices* **A** *except the* **0** *matrix.*

2. Other Matrix Operations

DEFINITION 4.9. *Let* **A** *be an* m *by* n *matrix, and let* r *be any real number. Then the scalar product* $r\mathbf{A}$ *is defined as the* m *by* n *matrix whose* ij^{th} *entry is* r *times the* ij^{th} *entry of* **A***:*

$$(r\mathbf{A})_{ij}=r(\mathbf{A})_{ij}$$

EXAMPLE 4.10. If

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

then

$$-2\mathbf{A} = \begin{bmatrix} -2 & 2 \\ -2 & -6 \end{bmatrix}$$

DEFINITION 4.11. *Let $\mathbf{A}$ and $\mathbf{B}$ be m by n matrices. Then the matrix sum $\mathbf{A}+\mathbf{B}$ is defined as the m by n matrix whose ij^{th} entry is the sum of the ij^{th} entries of $\mathbf{A}$ and $\mathbf{B}$:*

$$(\mathbf{A}+\mathbf{A})_{ij} = (\mathbf{A})_{ij} + (\mathbf{B})_{ij}$$

EXAMPLE 4.12. If

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

then

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 3 & 5 \end{bmatrix}$$

Combining these two operations of scalar multiplication and addition we can now from **linear combinations** of matrices; e.g. $2\mathbf{A} - 3\mathbf{B}$.

DEFINITION 4.13. *Let $\mathbf{A}$ be an m by n matrix, then the transpose $\mathbf{A}^T$ of $\mathbf{A}$ is the n by m such that*

$$\left(\mathbf{A}^T\right)_{ij} = (\mathbf{A})_{ji}$$

In other words, the entries in the i^{th} row of $\mathbf{A}^T$ are identical to the entries in the i^{th} column of $\mathbf{A}$.

EXAMPLE 4.14. If

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 1 \\ 3 & -1 \end{bmatrix}$$

then

$$\mathbf{A}^T = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$$

EXAMPLE 4.15. Recall that we can interprete an n-dimenional $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ either as a n by 1 matrix (which we called a column vector)

$$\mathbf{c} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

or as a 1 by n matrix (which we called a row vector)

$$\mathbf{r} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

Note that

$$\mathbf{c} = \mathbf{r}^T$$

and

$$\mathbf{r} = \mathbf{c}^T$$

DEFINITION 4.16. *An n by n matrix with the property that $\mathbf{A} = \mathbf{A}^T$ is called a **symmetric matrix**.*

EXAMPLE 4.17.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

is a symmetric matrix, but

$$\mathbf{B} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

is not symmetric because, for example

$$2 = (\mathbf{B})_{21} \neq \left(\mathbf{B}^T\right)_{21} \equiv (\mathbf{B})_{12} = 1$$

With a little experience it is easy to glance a matrix and determine whether or not it's symmetric.

THEOREM 4.18. *Suppose the matrix product* $\mathbf{AB}$ *is defined, then*

$$\begin{aligned} (\mathbf{A}^T)^T &= \mathbf{A} \\ (r\mathbf{A})^T &= r\mathbf{A}^T \\ (\mathbf{A}+\mathbf{B})^T &= \mathbf{A}^T + \mathbf{B}^T \\ (\mathbf{AB})^T &= \mathbf{B}^T\mathbf{A}^T \end{aligned}$$