LECTURE 2

The Geometry of Vector Spaces

1. Algebraic Definitions of Fundamental Geometric Objects

The purpose of this section is to establish the connections between our abstract algebraic formalism and the familar geometric objects of plane geometry. This is a crucial topic since virtually all of our intuitive understanding of linear algebra is based on geometrical visualizations. Luckily, however, the geometric objects and concepts required are fairly simple (if not not mundane.)

Definition 2.1. *We say that two vectors* $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *are* ***parallel*** *if there is a non-zero scalar* $r \in \mathbb{R}$ *such that* $\mathbf{w} = r\mathbf{v}$.

This definition is, of course, algebraic in nature. However, its geometrical implications are just what one expects: two vectors are parallel if and only their geometric representations point in the same or opposite direction.

We shall next present a discussion of the basic geometric objects (Euclidean) geometry: points, lines, and planes. To set this up, we begin with two abstract algebraic definitions.

Definition 2.2. *Given vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ *and scalars* $r_1, r_2, \ldots, r_k$ *we say that a vector of the form*

$$\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$$

is a ***linear combination*** *of the vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *with scalar coefficients* $r_1, r_2, \ldots, r_k$

Definition 2.3. *Given a set* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *of vectors in* $\mathbb{R}^n$*, the* ***span*** *of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *is the set of all possible linear combinations of the vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

$$\mathit{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k \mid r_1, r_2, \ldots, r_k \in \mathbb{R}\}$$

1.1. Lines.

There are two common geometrical ways of prescribing a straight line in a 3-dimensional space.

- Given two distinct points $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^3$, there is a unique line passing through both $\mathbf{p}_1$ and $\mathbf{p}_2$.
- Given one point $\mathbf{p}_0 \in \mathbb{R}^3$ and a direction $\mathbf{v}$, there is a unique line passing through $\mathbf{p}_0$ with the direction $\mathbf{v}$.

In this course, we shall think of a lines sets of points of the following form

$$\ell = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{p}_0 + \mathbf{v}t \quad , \quad t \in \mathbb{R}\} \tag{2.1}$$

The connection with the second geometrical description of a line is evident from the notation. To make the connection with the first geometrical description, all we have to do is set $\mathbf{p}_1 = \mathbf{p}_0 + \mathbf{v}$.

If we express the vectors $\mathbf{x}, \mathbf{p}_0$, and $\mathbf{v}$ in terms of components; e.g.

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, \ldots, x_n) \\ \mathbf{p}_0 &= (p_1, p_2, \ldots, p_n) \\ \mathbf{v} &= (v_1, v_2, \ldots, v_n) \end{aligned}$$

then we obtain from (2.1) the following *parametric equation* for a line

$$\begin{aligned} x_1 &= p_1 + v_1 t \\ x_2 &= p_2 + v_2 t \\ &\vdots \\ x_n &= p_n + v_n t \end{aligned}$$

A special, but very important, case of a line in $\mathbb{R}^n$ is the **span of a vector v**. According to the definition above this would be the set of vectors of the form

$$span\{\mathbf{v}\} = \{t\mathbf{v} \mid t \in \mathbb{R}\}$$

That is to say, it is a line passing through the origin ($\mathbf{p}_0 = \mathbf{0}$) in the direction $\mathbf{v}$.

1.2. Planes. Just as a line can be prescribed by specifying its direction and a single point on the line; a *plane* can be prescribed by specifying a single point $\mathbf{p}_0$ lying in the plane and two distinct directions $\mathbf{v}, \mathbf{u}$ lying in the plane. In vector notation such a prescription takes the form

$$\mathbf{P} = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{p}_0 + s\mathbf{u} + t\mathbf{v} \quad , \quad s, t \in \mathbb{R}\right\}$$

If we set

$$\begin{aligned} \mathbf{p}_0 &= (p_1, p_2, \ldots, p_n) \\ \mathbf{u} &= (u_1, u_2, \ldots, u_n) \\ \mathbf{v} &= (v_1, v_2, \ldots, v_n) \end{aligned}$$

then the relation $\mathbf{x} = \mathbf{p}_0 + \mathbf{u}s + \mathbf{v}t$ leads to the following *parametric representation* of the points in a plane

$$\begin{aligned} x_1 &= p_1 + u_1 s + v_1 t \\ x_2 &= p_2 + u_2 s + v_2 t \\ &\vdots \\ x_n &= p_n + u_n s + v_n t \end{aligned}$$

For general points $\mathbf{p}_0$ such planes do not in general pass through the origin. As in the case of lines, an important special case is when $\mathbf{p}_0 = \mathbf{0}$, so that the associated plane *does* pass through the origin. In this case, the plane may be thought of as a collection of vectors: indeed, it is the **span** of vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{P} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \mathbf{u}s + \mathbf{v}t \quad , \quad s, t \in \mathbb{R}\}$$

1.3. Hypersurfaces. It should now be pretty straightforward to generalize lines and planes to their analogs in higher dimensions.

If $\mathbf{p}_0$ is a point in $\mathbb{R}^n$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of *distinct* directions in $\mathbb{R}^n$ then the hypersurface passing through the point $\mathbf{p}_0$ in the directions $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is the set of points

$$\mathbf{P} = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \mathbf{p}_0 + s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k \quad ; \quad s_1, s_2, \ldots, s_k \in \mathbb{R}\right\}$$

Again, a case of special note is when $\mathbf{p}_0 = \mathbf{0}$ in which case $\mathbf{P}$ is the span of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

REMARK 2.4. At this juncture perhaps something should be said as to why the span of a set of vectors is such a special case of lines, planes, etc. The crucial point here is that if you have two vectors $\mathbf{u}$ and $\mathbf{w}$ contained in the span of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ then their sum, and in fact any linear combination of $\mathbf{u}$ and $\mathbf{w}$, will lie in the span of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. For

$$\begin{aligned} \mathbf{u} &\in span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \quad \Rightarrow \quad \mathbf{u} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k \\ \mathbf{w} &\in span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \quad \Rightarrow \quad \mathbf{w} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}k \end{aligned}$$

so

$$\mathbf{u}+\mathbf{w}=(r_1+s_1)\,\mathbf{v}_1+(r_2+s_2)\,\mathbf{v}_2+\cdots+(r_k+s_k)\,\mathbf{v}_k \quad \Rightarrow \quad \mathbf{u}+\mathbf{w}\in span\,\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$$

Consider now two points $\mathbf{p}_1=(1,2)$ and $\mathbf{p}_2=(3,1)$ on the line

$$\ell=\{(1,2)+(2,-1)t \mid t\in\mathbb{R}\}$$

We have

$$\mathbf{p}_1+\mathbf{p}_2=(4,3)$$

If this point lies on the line ℓ there must be a value of t such that

$$(4,3)=(1,2)+(2,-1)t \tag{2.2}$$

or

$$\begin{aligned} 4 &= 1+2t \\ 3 &= 2-t \end{aligned}$$

Solving the first equation for t yields

$$t=\frac{3}{2}$$

but solving the second equation for t yields

$$t=-1$$

Obviously, we cannot find a value of t such that (2.2) holds; so the vector sum of two points on a line is not, in general, another point on the line.

2. Geometrical Properties of Dot Products

THEOREM 2.5. ***(Cauchy-Schartz Inequality)*** *If* $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$, *then*

$$|\mathbf{u}\cdot\mathbf{v}|\leq|\mathbf{u}|\,|\mathbf{v}|$$

proof: Let $a=\mathbf{v}\cdot\mathbf{v}$ and $b=-\mathbf{u}\cdot\mathbf{v}$. If $a=0$ then $\mathbf{v}\cdot\mathbf{v}=0$, hence $\mathbf{v}=\mathbf{0}$ by the part 4 of Theorem 1.4 (Lecture 1). The inequality is thus trivially satisfied since both sides vanish identically when $\mathbf{v}=(0,0,0)$. Now suppose $a\neq 0$. By the Theorem 1.4 we have

$$\begin{aligned} 0 &\leq |a\mathbf{u}+b\mathbf{v}|=(a\mathbf{u}+b\mathbf{v})\cdot(a\mathbf{u}+b\mathbf{v}) \\ &= a^2\,(\mathbf{u}\cdot\mathbf{u})+2ab(\mathbf{u}\cdot\mathbf{v})+b^2(\mathbf{v}\cdot\mathbf{v}) \\ &= (\mathbf{v}\cdot\mathbf{v})^2\,(\mathbf{u}\cdot\mathbf{u})-2(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{v})^2+(\mathbf{u}\cdot\mathbf{v})^2(\mathbf{v}\cdot\mathbf{v}) \\ &= (\mathbf{v}\cdot\mathbf{v})^2\,(\mathbf{u}\cdot\mathbf{u})-(\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{v})^2 \end{aligned}$$

Dividing the extreme sides by $a=(\mathbf{v}\cdot\mathbf{v})$ (which is allowed since we assuming at this point that $a\neq 0$), we obtain

$$0\leq(\mathbf{v}\cdot\mathbf{v})\,(\mathbf{u}\cdot\mathbf{u})-(\mathbf{u}\cdot\mathbf{v})^2$$

or

$$(\mathbf{u}\cdot\mathbf{v})^2\leq(\mathbf{v}\cdot\mathbf{v})\,(\mathbf{u}\cdot\mathbf{u})=|\mathbf{v}|^2\,|\mathbf{u}|^2$$

Taking the positive square root of both sides now yields the desired inequality.

THEOREM 2.6. ***(Triangle Inequality)*** *If* $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$, *then*

$$|\mathbf{u}+\mathbf{v}|\leq|\mathbf{u}|+|\mathbf{v}|$$

proof: By the preceding theorem

$$\mathbf{u} \cdot \mathbf{v} \leq |\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}| \, |\mathbf{v}|$$

Thus,

$$|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2 \leq |\mathbf{u}|^2 + 2 \, |\mathbf{u}| \, |\mathbf{v}| + |\mathbf{v}|^2 = (|\mathbf{u}| + |\mathbf{v}|)^2$$

Taking the square root of both sides yields

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$