Math 2233 Sample Final
Summer 2017

1. Solve the following initial value problem

$$xy' + 3y = x^2 \qquad , \qquad y(1) = 1$$

- This is a first order linear ODE. Putting it in the standard form $y' + p(x)\,y = g(x)$, we have $p(x) = 3/x$ and $g(x) = x$. The integrating factor $\mu(x)$ is thus

$$\mu(x) = \exp\left(\int p(x)\,dx\right) = \exp\left(\int \frac{3}{x}dx\right) = \exp(3\ln|x|) = x^3$$

and the general solution is thus

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int g(x)\,\mu(x)\,dx + \frac{C}{\mu(x)} = \frac{1}{x^3}\int (x)\left(x^3\right)dx + \frac{C}{x^3} = \frac{1}{x^3}\left(\frac{1}{5}x^5\right) + \frac{C}{x^3} \\ &= \frac{1}{5}x^2 + \frac{C}{x^3} \end{aligned}$$

Plugging the general solution into the initial condition, we find

$$1 = y(1) = \frac{1}{5}(1)^2 + \frac{C}{(1)^3} = \frac{1}{5} + C \quad \Rightarrow \quad C = \frac{4}{5}$$

The solution to the initial value problem is thus

$$y(x) = \frac{1}{5}x^2 + \frac{4}{5x^3}$$

2.

(a) Use the Method of Variation of Parameters to find the general solution of

$$y'' - 4y = 3e^x \quad .$$

- We first need two independent solutions of the corresponding homogeneous problem $y'' - 4y = 0$. This is a Constant Coefficients ODE with characteristic equation

$$0 = \lambda^2 - 4 \quad \Rightarrow \quad \lambda = \pm 2 \quad \Rightarrow \quad y_1 = e^{2x} \quad , \quad y_2 = e^{-2x}$$

We have

$$W[y_1, y_2] = \left(e^{2x}\right)\left(-2e^{2x}\right) - \left(2e^{2x}\right)\left(e^{-2x}\right) = -2 - 2 = -4$$

and the Variation of Parameters formula gives us a particular solution of the inhomogeneous differential equation

$$\begin{aligned} y_p(x) &= -y_1\int \frac{y_2 g}{W[y_1,y_2]}dx + y_2\int \frac{y_1 g}{W[y_1,y_2]}dx = -e^{2x}\int \frac{\left(e^{-2x}\right)(3e^x)}{4}dx + e^{-2x}\int \frac{\left(e^{2x}\right)(3e^x)}{4}dx \\ &= -\frac{1}{4}e^{2x}\int e^{-x}dx + \frac{1}{4}e^{-2x}\int e^{3x}dx = -\frac{1}{4}\left(e^{2x}\right)\left(-e^{-x}\right) + \frac{1}{4}\left(e^{-2x}\right)\left(\frac{1}{3}e^{3x}\right) \\ &= \left(\frac{1}{4} + \frac{1}{12}\right)e^x = \frac{1}{3}e^x \end{aligned}$$

The general solution is thus

$$y(x) = \frac{1}{3}e^x + c_1 e^{2x} + c_2 e^{-2x}$$

3. Suppose $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$. Find a power series expression for $x^2 y''$.

- We first need to find the Tayloar expansion of x^2 about $x = 1$. If $f(x) = x^2$, then $f(1) = 1$, $f'(1) = 2x|_{x=1} = 2$, and $f''(1) = 2|_{x=1} = 2$. All higher derivatives vanish. Thus

$$x^2 = f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 1 + 2(x-1) + (x-1)^2$$

Using this identity we write

$$\begin{aligned}
x^2 y'' &= \left(1 + 2(x-1) + (x-1)^2\right) \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2} \\
&= \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=0}^{\infty} 2n(n-1) a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^n \\
&= \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=-1}^{\infty} (n+1)(n) a_{n+1} (x-1)^n + \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^n \\
&= 0 + 0 + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + 0 + \sum_{n=0}^{\infty} (n+1)(n) a_{n+1} (x-1)^n + \sum_{n=0}^{\infty} n(n-1) a_n (x-1)^n \\
&= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n(n+1) a_{n+1} + a_n\right] (x-1)^n
\end{aligned}$$

4. Determine the recursion relations for the power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ (about $x = 0$) for the following differential equation

$$y'' + xy' + 2y = 0$$

-

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} (n)(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2 a_n x^n \\
&= \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} -2 a_n x^n \\
&= 0 + 0 + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n \\
&= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+2) a_n\right] x^n
\end{aligned}$$

The recursion relations are obtained by setting the total coefficient of each x^n on the right equal to zero:

$$(n+2)(n+1) a_{n+2} + (n+2) a_n = 0 \quad , \quad n = 0, 1, 2, 3, \ldots$$

or

$$a_{n+2} = -\frac{a_n}{n+1} \quad , \quad n = 0, 1, 2, 3, \ldots$$

5. Consider the following initial value problem

$$xy'' - y = 0 \ , \ y(1) = 1 \ , \ y'(1) = 2$$

Given that the recursion relations for a power series solution of the form $\sum_{n=0}^{\infty} a_n (x-1)^n$ are

$$a_{n+2} = \frac{a_n - n(n+1) a_{n+1}}{(n+2)(n+1)}$$

write down (explicitly) the first four terms (i.e. up to order $(x-1)^3$) of this power series solution.

- The initial conditions require

$$\begin{aligned} a_0 &= y(1) = 1 \\ a_1 &= y'(1) = 2 \end{aligned}$$

The recursion relations will now furnish us with the rest of the coefficients:

$$\begin{aligned} a_2 &= a_{0+2} = \frac{a_0 - (0)(0+1)a_1}{(0+2)(0+1)} = \frac{a_0}{2} = \frac{1}{2} \\ a_3 &= a_{1+2} = \frac{a_1 - (1)(1+1)a_2}{(1+2)(1+1)} = \frac{1}{6}a_1 - \frac{2}{6}a_2 = \frac{2}{6} - \frac{2}{6}\left(\frac{1}{2}\right) = \frac{1}{6} \end{aligned}$$

Thus,

$$\begin{aligned} y(x) &= a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \cdots \\ &= 1 + 2(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \cdots \end{aligned}$$

6. Consider the differential equation $x^2(x+2)^2y'' + (x+2)y' + (x-1)y = 0$.

(a) Identify and classify the singular points of this differential equation.

- We have

$$\begin{aligned} p(x) &= \frac{x+2}{x^2(x+2)^2} = \frac{1}{x^2(x+2)} \quad \Rightarrow \quad \text{singular at } x = 0, -2 \\ q(x) &= \frac{x-1}{x^2(x+2)^2} \quad \Rightarrow \quad \text{singulr at } x = 0, -2 \end{aligned}$$

sing. pt.	$\deg(p, x_s)$	$\deg(q, x_s)$	type
$x_s = 0$	2	2	irregular
$x_s = -2$	1	2	regular

(b) What is the minimal radius of convergence of a power series solution of this equation about the point $x = 4$.

- The singular point $x = 0$ is closed to the expansion point $x_0 = 4$ and the distance between these points is 4. Therefore,

$$R = 4.$$

7. Find a function having the following Laplace transform:

$$\frac{7}{s^2+9} + \frac{1}{s^5}$$

-

$$\begin{aligned} \frac{7}{s^2+9} + \frac{1}{s^5} &= \frac{7}{3}\frac{3}{(s-0)^2+3^2} + \frac{1}{4!}\frac{4!}{s^{4+1}} \\ &= \frac{7}{3}\mathcal{L}\left[e^{0x}\sin(3x)\right] + \frac{1}{24}\mathcal{L}\left[x^4\right] \\ &= \mathcal{L}\left[\frac{7}{3}\sin(3x) + \frac{1}{24}x^4\right] \\ \Rightarrow \quad & f(x) = \frac{7}{3}\sin(3x) + \frac{1}{24}x^4 \end{aligned}$$

8. Solve the following differential equation using Laplace transforms:

$$y'' + 2y' - 3y = 0 \quad ; \quad y(0) = 1 \quad , \quad y'(0) = 0 \quad .$$

- Taking the Laplace transform of the differential equation:

$$\left(s^2\mathcal{L}[y]\right) - sy(0) - y'(0) + 2(s\mathcal{L}[y] - y(0)) - 3\mathcal{L}[y] = 0$$

or

$$\left(s^2 + 2s - 3\right)\mathcal{L}[y] = sy(0) + y'(0) + 2y(0) = s + 2$$

or

$$\mathcal{L}[y] = \frac{s+2}{s^2+2s-3} = \frac{s+2}{(s+3)(s-1)}$$

Now, by Partial Fractions,

$$\frac{s+2}{(s+3)(x-1)} = \frac{A}{s+3} + \frac{B}{s-1} \quad \Rightarrow \quad s + 2 = A(s-1) + B(s+3)$$

Evaluating the last equation at $s = 1$ yields, $2 = A(0) + B(4) \quad \Rightarrow \quad B = 1/2$. Evaluating it at $s = -3$, yields $-1 = A(-4) + B(0) \quad \Rightarrow \quad A = \frac{1}{4}$. Thus,

$$\mathcal{L}[y] = \frac{1}{4}\frac{1}{s+3} + \frac{1}{2}\frac{1}{s-1} = \frac{1}{4}\mathcal{L}\left[e^{-3x}\right] + \frac{1}{2}\mathcal{L}[e^x] = \mathcal{L}\left[\frac{1}{4}e^{-3x} + \frac{1}{2}e^x\right]$$

Thus,

$$y(x) = \frac{1}{4}e^{-3x} + \frac{1}{2}e^x$$