Math 2233
Homework Set 9

1. Compute the Laplace transform of the following functions.

(a) $f(t) = t$

- Let $f(t) = t$.

$$\mathcal{L}[f] = \int_0^\infty te^{-t}dt$$

Integrating by parts, with

$$\begin{aligned} u &= t & dv &= e^{-st}dt \\ du &= dt & v &= -\tfrac{1}{s}e^{-st} \end{aligned}$$

we get

$$\begin{aligned} \int_0^\infty te^{-t}dt &= \int_0^\infty vdu \\ &= uv|_0^\infty - \int_0^\infty vdu \\ &= (t)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty \left(-\frac{1}{s}e^{-st}\right)dt \\ &= 0 - 0 - \frac{1}{s^2}e^{-st}\Big|_0^\infty \\ &= \frac{1}{s^2} \end{aligned}$$

□

(b) $f(t) = t^n$

- Now let $f(t) = t^n$.

$$\mathcal{L}[f] = \int_0^\infty t^n e^{-t}dt$$

Integrating by parts, with

$$\begin{aligned} u &= t^n & dv &= e^{-st}dt \\ du &= nt^{n-1}dt & v &= -\tfrac{1}{s}e^{-st} \end{aligned}$$

we get

$$\begin{aligned}
\int_0^\infty t^n e^{-t} dt &= \int_0^\infty v du \\
&= uv|_0^\infty - \int_0^\infty v du \\
&= (t^n)\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty \left(-\frac{1}{s}e^{-st}\right)\left(nt^{n-1}dt\right) \\
&= 0 - 0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt \\
&= \frac{n}{s}\mathcal{L}\left[t^{n-1}\right] \\
&= \frac{n}{s}\frac{n-1}{s}\mathcal{L}\left[t^{n-2}\right] \\
&= \frac{n(n-1)(n-2)}{s^3}\mathcal{L}\left[t^{n-3}\right] \\
&\vdots \\
&= \frac{n(n-2)\cdots(2)}{s^{n-1}}\mathcal{L}[t] \\
&= \frac{n!}{s^{n+1}}
\end{aligned}$$

□

2. Use the formula

$$\sin(bt) = \frac{e^{ibt} - e^{-ibt}}{2i}$$

to compute the Laplace transform of $\sin(bt)$.

- Let

$$f(t) = \sin(bt) = \frac{1}{2i}\left(e^{ibt} - e^{-ibt}\right) \quad .$$

Then

$$\begin{aligned}
\mathcal{L}[\sin(bt)] &= \int_0^\infty \frac{1}{2i}\left(e^{ibt} - e^{-ibt}\right)e^{-st}dt \\
&= \frac{1}{2i}\int_0^\infty \left(e^{(-s+ib)t} - e^{-(s+ib)t}\right) dt \\
&= \frac{1}{2i}\left(\frac{1}{-s+ib}e^{-st}e^{ibt} + \frac{1}{s+ib}e^{-st}e^{-ibt}\right)\Big|_0^\infty \\
&= \frac{1}{2i}\left(\frac{1}{s-ib} - \frac{1}{s+ib}\right) \\
&= \frac{1}{2i}\left(\frac{s+ib-(s-ib)}{s^2+b^2}\right) \\
&= \frac{b}{s^2+b^2}
\end{aligned}$$

□

3. Invert the following Laplace transforms.

(a) $\mathcal{L}[f] = \dfrac{3}{s^2+4}$

- Note that the denominator is a sum of squares:

$$\begin{aligned} \frac{3}{s^2+4} &= 3\frac{1}{s^2+(2)^2} = \left(\frac{3}{2}\right)\frac{2}{s^2+(2)^2} = \left(\frac{3}{2}\right)\mathcal{L}\left[\sin(2x)\right] = \mathcal{L}\left[\frac{3}{2}\sin(2x)\right] \\ &\Rightarrow \quad f(x) = \frac{3}{2}\sin(2x) \end{aligned}$$

□

(b) $\mathcal{L}[f] = \dfrac{2}{s^2+3s-4}$

- $$\frac{2}{s^2+3s-4} = \frac{2}{(s+4)(s-1)}$$

  Since the denominator is easily factorized, we'll try a partial fractions expansion.

$$\begin{aligned} \frac{2}{(s+4)(s-1)} &= \frac{A}{s+4} + \frac{B}{s-1} \quad \Rightarrow \quad 2 = A(s-1) + B(s+4) \\ s &= 1 \quad \Rightarrow \quad 2 = A(0) + B(5) \quad \Rightarrow \quad B = \frac{2}{5} \\ s &= -4 \quad \Rightarrow \quad 2 = A(-5) + B(0) \quad \Rightarrow \quad A = -\frac{2}{5} \end{aligned}$$

  So now

$$\begin{aligned} \mathcal{L}[f] &= \frac{2}{s^2+3s-4} = \frac{2}{(s+4)(s-1)} = -\frac{2}{5}\left(\frac{1}{s+4}\right) + \frac{2}{5}\left(\frac{1}{s-1}\right) \\ &= -\frac{2}{5}\mathcal{L}\left[e^{-4x}\right] + \frac{2}{5}\mathcal{L}\left[e^{x}\right] = \mathcal{L}\left[-\frac{2}{5}e^{-4x} + \frac{2}{5}e^{x}\right] \\ &\Rightarrow \quad f(x) = -\frac{2}{5}e^{-4x} + \frac{2}{5}e^{x} \end{aligned}$$

□

(c) $\mathcal{L}[f] = \dfrac{2s+2}{s^2+2s+5}$

- – In this problem, the denominator does not factorize easily, but it can be written as a sum of squares:

$$\frac{2s+2}{s^2+2s+5} = \frac{2s+2}{s^2+2s+1+(2)^2} = \frac{2s+2}{(s+1)^2+(2)^2}$$

  and so we'll try to realize it as a linear combination of the Laplace transforms of the form

$$\mathcal{L}\left[e^{at}\cos(bt)\right] = \frac{s-a}{(s-a)^2+b^2} \quad , \quad \mathcal{L}\left[e^{at}\sin(bt)\right] = \frac{b}{(s-a)^2+b^2}$$

  We have

$$\begin{aligned} \mathcal{L}[f] &= \frac{2s+2}{(s+1)^2+(2)^2} = 2\left(\frac{s+1}{(s+1)^2+(2)^2}\right) = 2\mathcal{L}\left[e^{-x}\cos(2x)\right] = \mathcal{L}\left[2e^{-x}\cos(2x)\right] \\ &\Rightarrow \quad f(x) = 2e^{-x}\cos(2x) \end{aligned}$$

□

(d) $\mathcal{L}[f] = \dfrac{2s+1}{s^2-2s+2}$

- Again the denominator does not factorize easily, so we'll try to express the denominator as a sum or difference of squares.

$$\begin{aligned}
\mathcal{L}[f] &= \frac{2s+1}{s^2-2s+2} = \frac{2s+1}{s^2-2s+1+1} = \frac{2s+1}{(s-1)^2+(1)^2} = \frac{2s-2+2+1}{(s-1)^2+(1)^2} = \frac{2(s-1)+3}{(s-1)^2+(1)^2} \\
&= 2\left(\frac{s-1}{(s-1)^2+(1)^2}\right) + 3\left(\frac{1}{(s-1)^2+(1)^2}\right) = 2\mathcal{L}[e^x\cos(x)] + 3\mathcal{L}[e^x\sin(x)] \\
&= \mathcal{L}[2e^x\cos(x) + 3e^x\sin(x)] \\
&\Rightarrow \quad f(x) = 2e^x\cos(x) + 3e^x\sin(x)
\end{aligned}$$

□

(e) $\mathcal{L}[f] = \dfrac{1-2s}{s^2+4s+5}$

- The denominator does not factorize easily, so we'll try to first try to express it a sum or difference of squares.

$$\begin{aligned}
\mathcal{L}[f] &= \frac{1-2s}{s^2+4s+5} = \frac{1-2s}{s^2+4s+4+1} = \frac{1-2s}{(s+2)^2+1} = \frac{1-2s-4+4}{(s+2)^2+1^2} \\
&= \frac{5-2(s+2)}{(s+2)^2+1^2} = 5\left(\frac{1}{(s+2)^2+(1)^2}\right) - 2\left(\frac{s+2}{(s+2)^2+(1)^2}\right) \\
&= 5\mathcal{L}\left[e^{-2x}\sin(x)\right] - 2\mathcal{L}\left[e^{-2x}\cos(x)\right] = \mathcal{L}\left[5e^{-2x}\sin(x) - 2e^{-2x}\cos(x)\right] \\
&\Rightarrow \quad f(x) = 5e^{-2x}\sin(x) - 2e^{-2x}\cos(x)
\end{aligned}$$

□

4. Use the Laplace transform to solve the given initial value problems.

(a) $y'' - y' - 6y = 0 \quad ; \quad y(0) = 1 \quad , \quad y'(0) = -1$

- Taking the Laplace transform of both sides of the differential equation yields

$$\begin{aligned}
0 &= \mathcal{L}[y''] - \mathcal{L}[y'] - \mathcal{L}[6y] \\
&= \left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) - (s\mathcal{L}[y] - y(0)) - 6\mathcal{L}[y] \\
&= s^2\mathcal{L}[y] - s(1) - (-1) - s\mathcal{L}[y] + (1) - 6\mathcal{L}[y] \\
&= \left(s^2 - s - 6\right)\mathcal{L}[y] - s + 2
\end{aligned}$$

or

$$\mathcal{L}[y] = \frac{s-2}{s^2-s-6} = \frac{s-2}{(s+2)(s-3)}$$

the differential equation for y becomes an algebraic equation for $\mathcal{L}[y]$. To undo this Laplace transform we first carry out a partial fractions expansion of the right hand side of the equation for $\mathcal{L}[y]$.

$$\frac{s-2}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3} \Rightarrow \quad s-2 = A(s-3) + B(s+2)$$

This expansion must be valid for all values of s; in particular when $s = -2$ and when $s = 3$. In the former case

we have

$$s = -2 \quad \Rightarrow \quad -4 = (-2) - 2 = A(-2-3) + B(-2+2) = -5A$$

so we must have $A = \frac{4}{5}$. In the latter case, we have

$$s = 3 \quad \Rightarrow \quad 1 = (3) - 2 = A(3-3) + B(3+2) = 5B$$

so $B = \frac{1}{5}$. We then have

$$\begin{aligned}
\mathcal{L}[y] &= \frac{s-2}{(s+2)(s-3)} \\
&= \frac{4}{5}\frac{1}{s+2} + \frac{1}{5}\frac{1}{s-3} \\
&= \frac{4}{5}\mathcal{L}[e^{-2x}] + \frac{1}{5}\mathcal{L}[e^{3x}] \\
&= \mathcal{L}\left[\frac{4}{5}e^{-2x} + \frac{1}{5}e^{3x}\right]
\end{aligned}$$

Hence, (taking inverse Laplace transform of both sides)

$$y = \frac{4}{5}e^{-2x} + \frac{1}{5}e^{3x}$$

□

(b) $y'' - 2y' + 2y = 0 \quad ; \quad y(0) = 0 \quad , \quad y'(0) = 1$

- Taking the Laplace transform of both sides of the differential equation yields

$$\begin{aligned}
0 &= s^2\mathcal{L}[y] - sy(0) - y'(0) - 2\left(s\mathcal{L}[y] - y(0)\right) + 2\mathcal{L}[y] \\
&= \left(s^2 - 2s + 2\right)\mathcal{L}[y] - 1
\end{aligned}$$

or

$$\mathcal{L}[y] = \frac{1}{s^2 - 2s + 2} = \frac{1}{s^2 - 2s + 1 + 1} = \frac{1}{(s-1)^2 + 1}$$

We now consult a table of Laplace transform and spot the following identity

$$\mathcal{L}\left[e^{at}\sin(bt)\right] = \frac{b}{(s-a)^2 + b^2}$$

which looks just like the right hand side of our expression for $\mathcal{L}[y]$ once we thake $a = 1$ and $b = 1$. We conclude

$$\mathcal{L}[y] = \mathcal{L}[e^x \sin(x)]$$

or

$$y(x) = e^x \sin(x)$$

□

$y'' - 2y' - 2y = 0 \quad ; \quad y(0) = 2 \quad , \quad y'(0) = 0$

- Taking the Laplace transform of the differential equation we get

$$\begin{aligned}
0 &= s^2\mathcal{L}[y] - sy(0) - y'(0) - 2\left(s\mathcal{L}[y] - y(0)\right) - 2\mathcal{L}[y] \\
&= \left(s^2 - 2s - 2\right)\mathcal{L}[y] - 2s + 4 \\
&= \left(s^2 - 2s - 2\right)\mathcal{L}[y] - 2s + 4
\end{aligned}$$

Thus,

$$\mathcal{L}[y] = \frac{2s-4}{s^2 - 2s - 2}$$

Note that the denominator is easily written as a difference of squares; viz.,

$$\begin{aligned}
\mathcal{L}[y] &= \frac{2s-4}{s^2 - 2s - 2} \\
&= 2\frac{s-2}{s^2 - 2s + 1 - 3} \\
&= 2\frac{s-2}{(s-1)^2 - \left(\sqrt{3}\right)^2}
\end{aligned}$$

so to invert this Laplace transform, the following Laplace transforms (from the table of Laplace transforms) might be useful.

$$\mathcal{L}[e^{at}\cosh(bt)] = \frac{s-a}{(s-a)^2-b^2}$$
$$\mathcal{L}[e^{at}\sinh(bt)] = \frac{b}{(s-a)^2-b^2}$$

We have

$$\begin{aligned}\mathcal{L}[y] &= 2\frac{s-2}{(s-1)^2-\left(\sqrt{3}\right)^2} = 2\frac{(s-1)-1}{(s-1)^2-\left(\sqrt{3}\right)^2} \\ &= 2\frac{s-1}{(s-1)^2-\left(\sqrt{3}\right)^2} - 2\frac{1}{(s-1)^2-\left(\sqrt{3}\right)^2} \\ &= 2\frac{s-1}{(s-1)^2-\left(\sqrt{3}\right)^2} - \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{(s-1)^2-\left(\sqrt{3}\right)^2}\end{aligned}$$

So taking $a = 1$ and $b = \sqrt{3}$ and using the two Laplace transforms above, we have

$$\begin{aligned}\mathcal{L}[y] &= 2\mathcal{L}[e^x\cosh(\sqrt{3}x)] - \frac{2}{\sqrt{3}}\mathcal{L}\left[e^x\sinh(\sqrt{3}x)\right] \\ &= \mathcal{L}\left[2e^x\cosh(\sqrt{3}x) - \frac{2}{\sqrt{3}}e^x\sinh(\sqrt{3}x)\right]\end{aligned}$$

so

$$y(x) = 2e^x\cosh(\sqrt{3}x) - \frac{2}{\sqrt{3}}e^x\sinh(\sqrt{3}x)$$

□

(d) $y'' + 2y' + y = 4e^{-t}$; $y(0) = 2$, $y'(0) = -1$

- Taking the Laplace transform of both sides of the differential equation we get

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2\left(s\mathcal{L}[y] - y(0)\right) + \mathcal{L}[y] = \mathcal{L}[4e^{-t}]$$

or

$$\left(s^2 + 2s + 1\right)\mathcal{L}[y] - 2s + 1 - 4 = \frac{4}{s+1}$$

or

$$(s+1)^2\mathcal{L}[y] = \frac{4}{s+1} + 2s + 3 = \frac{4 + 2s^2 + 3s + 3}{s+1} = \frac{2s^2 + 5s + 7}{s+1}$$

or

$$\mathcal{L}[y] = \frac{2s^2 + 5s + 7}{(s+1)^3}$$

We now determine the partial fractions expansion of the right hand side. The general *ansatz* is

$$\frac{P(x)}{(s+a)^3} = \frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{(s+a)^3}$$

and so we will try to find constants A, B, C such that

$$\frac{2s^2 + 5s + 7}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}.$$

Multiplying both sides by $(s+1)^3$ we get

$$2s^2 + 5s + 7 = A(s+1)^2 + B(s+1) + C \quad .$$

Plugging in $s = -1$ we find

$$2 - 5 + 7 = C$$

or

$$C = 4.$$

Plugging in $s = 0$ yields

$$7 = A + B + C = A + B + 4$$

or

$$A + B = 3.$$

Plugging in $s = 1$ yields

$$14 = 4A + 2B + C = 4A + 2B + 4$$

or

$$4A + 2B = 10$$

or

$$2A + B = 5.$$

We now solve

$$\begin{aligned} A + B &= 3 \\ 2A + B &= 5 \end{aligned}$$

for A and B. Subtracting the first equation from the second we obtain

$$A + 0 = 2 \qquad \Rightarrow \qquad A = 2.$$

Now the first equation yields

$$2 + B = 3 \qquad \Rightarrow \qquad B = 1.$$

Thus, $A = 2$, $B = 1$, and $C = 4$. Applying this partial fractions expansion to the equation for $\mathcal{L}[y]$ now yields

$$\mathcal{L}[y] = \frac{2s^2 + 3s + 7}{(s+1)^3} = \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{4}{(s+1)^3}$$

Now from a Table of Laplace transforms we find

$$\mathcal{L}\left[t^n e^{at}\right] = \frac{n!}{(s+a)^{n+1}}$$

Hence

$$\begin{aligned} \frac{1}{s+1} &= \mathcal{L}[e^{-t}] \\ \frac{1}{(s+1)^2} &= \mathcal{L}[te^{-t}] \\ \frac{1}{(s+1)^3} &= \frac{1}{2}\mathcal{L}[t^2 e^{-t}] \end{aligned}$$

so

$$\begin{aligned} \mathcal{L}[y] &= 2\mathcal{L}[e^{-t}] + 1\mathcal{L}[te^{-t}] + 4\frac{1}{2}\mathcal{L}[t^2 e^{-t}] \\ &= \mathcal{L}\left[2e^{-t} + te^{-t} + 2t^2 e^{-t}\right] \end{aligned}$$

Taking the inverse Laplace transform of both sides we finally get

$$y(t) = \left(2t^2 + t + 2\right) e^{-t}.$$

□