Math 2233
Homework Set 8

1. Determine the lower bound for the radius of convergence of series solutions about each given point x_o.

(a) $y'' + 4y' + 6xy = 0$, $x_0 = 0$

- Since the coefficient functions

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 6x \end{aligned}$$

are perfectly analytic for all x, the differential equation thus possesses no singular points. Thus, every power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

will converges for all x and all x_o. In particular, the radius of convergence for solutions about $x_o = 0$ will be infinite. □

(b) $(x-1)y'' + xy' + 6xy = 0$, $x_0 = 4$

- Since the coefficient functions

$$\begin{aligned} p(x) &= \frac{x}{x-1} \\ q(x) &= \frac{6x}{x-1} \end{aligned}$$

are both undefined for $x = 1$. Therefore, $x = 1$ is a singular point for this differential equation. According to the theorem stated in lecture, if

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

is a power series solution, then its radius of convergence will be at least as large as the distance (in the complex plane) from the expansion point x_o and the closest singularity of the functions $p(x)$ and $q(x)$. In the case at hand, $x_0 = 4$ and the closest (in fact, the only) singular point of the coefficient functions $p(x)$ and $q(x)$ is $x = 1$. Since

$$\|4 - 1\| = 3$$

we can conclude that the radius of convergence of a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-4)^n$$

will be at least 3. In other words, the series solution will be valide for all x in the interval

$$|x - 4| < 3$$

or, equivalently, for all x such that

$$1 < x < 7$$

□

(c) $\left(4 + x^2\right) y'' + 4xy' + y = 0$, $x_0 = 0$

- In this case, the coefficient functions

$$\begin{aligned} p(x) &= \frac{4x}{4+x^2} \\ q(x) &= \frac{1}{4+x^2} \end{aligned}$$

both have singularities when

$$4 + x^2 = 0 \quad \Rightarrow \quad x = \pm 2i$$

These two singularies correspond to the points $(0, \pm 2)$ when we represent points in the complex plane as points in the two dimenional plane. Under this representation of the complex plane, the expansion point $x_0 = 0$ corresponds to the point $(0,0)$. Therefore the distances between the expansion point and the singularity are

$$\begin{aligned} dist(2i,0) &= \sqrt{(0-0)^2+(2-0)^2} = 2 \\ dist(-2i,0) &= \sqrt{(0-0)^2+(-2-0)^2} = 2 \end{aligned}$$

Hence, the minimal distance is 2, and so the radius of convergence of a power series solution about 0 is at least 2. □

(d) $\left(1+x^2\right)y'' + 4xy' + y = 0$, $x_0 = 2$

- In this case the coefficient functions

$$\begin{aligned} p(x) &= \frac{4x}{1+x^2} \\ q(x) &= \frac{1}{1+x^2} \end{aligned}$$

both have singularities when

$$1 + x^2 = 0 \quad \Rightarrow \quad x = \pm i$$

These two singularies correspond to the points $(0, \pm 1)$ when we represent points in the complex plane as points in the two dimenional plane. Under this representation of the complex plane, the expansion point $x_0 = 2$ corresponds to the point $(2,0)$. Therefore the distances between the expansion point and the singularity are

$$\begin{aligned} dist(i,2) &= \sqrt{(0-2)^2+(1-0)^2} = \sqrt{5} \\ dist(-i,2) &= (0-2)^2+(-1-0)^2 = \sqrt{5} \end{aligned}$$

Hence, the distance between the expansion point and the closest singularity is $\sqrt{5}$ and so the radius of convergence of a power series solution about the point $x_o = 2$ will be at least $\sqrt{5}$. □

2. Determine the singular points of the following differential equations and state whether they are regular or irregular singular points.

(a) $xy'' + (1-x)y' + xy = 0$

- In this case, the coefficient functions are

$$\begin{aligned} p(x) &= \frac{1-x}{x} \\ q(x) &= 1 \end{aligned}$$

Since $p(x)$ is undefined for $x = 0$, 0 is a singular point. Since the limits

$$\begin{aligned} \lim_{x\to 0}(x-0)p(x) &= \lim_{x\to 0}(1-x) = 1 \\ \lim_{x\to 0}(x-0)^2 q(x) &= \lim_{x\to 0} x^3 = 0 \end{aligned}$$

both exist, $x = 0$ is a regular singular point. Alternatively, one could say that because the degree of the singularity of the function $p(x)$ at the point $x = 0$ is less than or equal to 1 and the degree of the singularity of the function $q(x)$ is less than or equal to 2, we have regular singular point at $x = 0$. □

(b) $x^2(1-x)^2y'' + 2xy + 4y = 0$

- In this case, the coefficient functions are

$$\begin{aligned} p(x) &= \frac{2}{x(1-x)^2} \\ q(x) &= \frac{4}{x^2(1-x)^2} \end{aligned}$$

The function $p(x)$ evidently has a singularity of degree 1 at $x = 0$ and a singularity of degree 2 at $x = 1$. The function $q(x)$ has singularities of degree 2 at $x = 0$ and $x = 1$. In order to be a regularity singular point the degree of the singularity of $p(x)$ must not exceed 1 and the degree of the singularity of $q(x)$ must not exceed 2. Therefore, $x = 0$ is a regular singular point and $x = 1$ is an irregular singular point.. □

(c) $(1-x^2)^2 y'' + x(1-x)y' + (1+x)y = 0$

- In this case,the coefficient functions are

$$\begin{aligned} p(x) &= \frac{x(1-x)}{(1-x^2)^2} = \frac{x(1-x)}{(1-x)^2(1+x)^2} = \frac{x}{(1-x)(1+x)^2} \\ q(x) &= \frac{1+x}{(1-x)^2(1+x)^2} = \frac{1}{(1-x)^2(1+x)} \end{aligned}$$

The function $p(x)$ evidently has a singularity of degree 1 at $x = 1$ and a singularity of degree 2 at $x = -1$. The function $q(x)$ has a singularity of degree 1 at $x = 1$ and a singularity of degree 2 at $x = -1$. In order to be a regularity singular point the degree of the singularity of $p(x)$ must not exceed 1 and the degree of the singularity of $q(x)$ must not exceed 2. Therefore, $x = 1$ is a regular singular point and $x = -1$ is an irregular singular point. □

3. The following differential equation has a regular singular point at $x = 0$. Determine the indicial equations, the roots of the indicial equations, the recursion relations, and the first four terms of two linearly independent series solutions.

$$2xy'' + y' + xy = 0 \quad .$$

- We make the ansatz

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \qquad , \quad a_0 \neq 0 \qquad ,$$

and plug into the differential equation to obtain

$$\begin{aligned} 0 &= 2x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \quad + x \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0} a_n x^{n+r+1} \\ &= \sum_{n=0}^{\infty} \left(2(n+r)(n+r-1) + (n+r)\right) a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} \\ &= \left(2r(r-1) + r\right) a_0 x^{r-1} + \left(2(1+r)r + 1 + r\right) a_1 x^r + \sum_{n=2}^{\infty} \left(2(n+r)(n+r-1) + (n+r)\right) a_n x^{n+r-1} \\ &\quad + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} \\ &= \left(2r^2 - r\right) a_0 x^{r-1} + \left(2r^2 + 3r + 1\right) a_1 x^r + \sum_{n=2}^{\infty} \left[\left(2(n+r)(n+r-1) + (n+r)\right) a_n + a_{n-2}\right] x^{n+r-1} \\ &= \left(2r^2 - r\right) a_0 x^{r-1} + \left(2r^2 + 3r + 1\right) a_1 x^r + \sum_{n=2}^{\infty} \left[\left(2(n+r)^2 - (n+r)\right) a_n + a_{n-2}\right] x^{n+r-1} \end{aligned}$$

Setting the total coefficient of x^{r-1}, x^r, and x^{n+r-1} equal to zero we obtain the following equations

$$\begin{aligned} 0 &= (2r-1)\, r a_0 \\ 0 &= (2r+1)\,(r+1))\, a_1 \\ a_n &= \frac{-a_{n-2}}{2(n+r)^2-(n+r)} \qquad n = 2,3,4,\ldots \end{aligned}$$

Since a_0 is assumed to be non-zero the first equation leads to

$$0 = r(2r-1) \qquad \Rightarrow \qquad r = 0, \frac{1}{2} \quad .$$

If $r = 0$, then the second equation produces

$$0 = (0+1)(0+1)a_1 = a_1 \qquad \Rightarrow \qquad a_1 = 0.$$

The third equation furnishes recursion relations that allow us to express all the even coefficients a_{2i} in terms of a_0 and all the odd coefficients in terms of a_1. However, because $a_1 = 0$ only even powers of x will occur.

To see this, let us first take $r = 0$. Then the recursion relation is

$$a_n = \frac{-a_{n-2}}{2n^2 - n}$$

so

$$\begin{aligned} a_2 &= \frac{a_0}{8-2} = \frac{-a_0}{6} \\ a_3 &= \frac{-a_1}{18-3} = 0 \\ a_4 &= \frac{-a_2}{32-4} = \frac{a_0}{168} \\ a_5 &= \frac{-a_3}{50-5} = 0 \\ &\vdots \end{aligned}$$

Thus to order x^5 one solution will be

$$y_1(x) = a_0 \left(1 - \frac{1}{6}x^2 + \frac{1}{192}x^4 + \cdots\right)$$

To get a second linearly independent solution we solve the recursion relations when $r = \frac{1}{2}$:

$$a_n = \frac{-a_{n-2}}{2\left(n+\frac{1}{2}\right)^2 + \left(n+\frac{1}{2}\right)} = \frac{-2a_{n-2}}{(2n+1)^2 - (2n+1)}$$

$$\begin{aligned} a_2 &= \frac{-2a_0}{25-5} = -\frac{1}{10}a_0 \\ a_3 &= \frac{-2a_1}{49-7} = 0 \\ a_4 &= \frac{-2a_2}{81-9} = \frac{1}{360}a_0 \\ a_5 &= \frac{-2a_3}{121-11} = 0 \\ &\vdots \end{aligned}$$

So we also have a solution (up to order x^5)

$$y_2 = a_0 x^{1/2} \left(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 + \cdots\right)$$

$\square$

4.The following differential equations have a regular singular point at $x = 0$. Determine the indicial equation and the recursion relations corresponding to the largest root of the indicial equation. Write down the first four terms of the corresponding series expansion.

(a) $xy'' + y = 0$

- This differential equation has a regular singular point at $x = 0$. Setting

$$y(x) = \sum_{n=0} a_n x^{n+r} \qquad , \quad a_0 \neq 0 \quad ,$$

and plugging into the differential equation we get

$$\begin{aligned} 0 &= x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} \end{aligned}$$

In order to combine the two series on the right we first shift the summation on the first by

$$k + r = n + r - 1 \qquad \Rightarrow \qquad \begin{array}{l} k = n-1 \\ n = k+1 \end{array}$$

to obtain

$$\begin{aligned} 0 &= \sum_{k=-1}^{\infty}(k+1+r)(k+r)a_{k+1}x^{k+r} + \sum_{k=0}^{\infty} a_k x^{k+r} \\ &= (-1+1+r)(-1+r)\,a_{-1+1}x^{-1+r} + \sum_{k=-1}^{\infty}(k+1+r)(k+r)a_{k+1}x^{k+r} \\ &\qquad + \sum_{k=0}^{\infty} a_k x^{k+r} \\ &= r\,(r-1)\,a_0 x^{r-1} + \sum_{k=0}^{\infty}\left((k+r+1)(k+r)a_{k+1} + a_k\right)x^{k+r} \end{aligned}$$

Demanding that the total coefficient of each power of x vanish we thus obtain

$$0 = r(r-1)a_0 \qquad \Rightarrow \qquad r = 0, 1$$

$$0 = (k+r+1)(k+r)a_{k+1} + a_k \qquad \Rightarrow \qquad a_{k+1} = \frac{-a_k}{(k+r+1)(k+r)}$$

Noting that the two roots of the indicial equation $r(r-1) = 0$ differ only by an integer, we follow the instructions in the statement of the problem and look for a solution corresponding to the larger root $r = 1$.
For this value of r the recursion relations are

$$a_{k+1} = \frac{-a_k}{(k+2)(k+1)}$$

Thus,

$$\begin{aligned} a_1 &= \frac{-a_0}{(2)(1)} = -\frac{a_0}{2} \\ a_2 &= \frac{-a_1}{(3)(2)} = \frac{a_0}{(3)(2)(2)} = \frac{a_0}{12} \\ a_3 &= \frac{-a_2}{(4)(3)} = \frac{-a_0}{(4)(3)(12)} = -\frac{a_0}{144} \end{aligned}$$

Thus the first four terms of this series solution will be

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots \\ &= a_0 x - \frac{a_0}{2} x^2 + \frac{a_0}{12} x^3 - \frac{a_0}{144} x^4 \\ &= a_0 \left(x - \frac{1}{2} x^2 + \frac{1}{12} x^3 - \frac{1}{144} x^4 + \cdots \right) \end{aligned}$$

$\square$

(b) $xy'' + (1-x)y' - y = 0$

- Setting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

and plugging in we obtain

$$\begin{aligned} 0 &= x \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + (1-x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ &\qquad - \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ &\qquad - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= \sum_{k=-1}^{\infty} (k+1+r)(k+r) a_{k+1} x^{k+r} + \sum_{k=-1}^{\infty} (k+1+r) a_{k+1} x^{k+r} \\ &\qquad - \sum_{k=0}^{\infty} (k+r) a_k x^{k+r} - \sum_{k=0}^{\infty} a_k x^{k+r} \\ &= r(r-1) a_0 x^{r-1} + \sum_{k=0}^{\infty} (k+1+r)(k+r) a_{k+1} x^{k+r} \\ &\qquad + r a_0 x^{r-1} + \sum_{k=0}^{\infty} (k+1+r) a_{k+1} x^{k+r} \\ &\qquad - \sum_{k=0}^{\infty} (k+r) a_k x^{k+r} - \sum_{k=0}^{\infty} a_k x^{k+r} \\ &= r^2 a_0 x^{r-1} \\ &\quad + \sum_{k=0}^{\infty} \left(\left((k+r+1)(k+r) + (k+r+1) \right) a_{k+1} - \left((k+r) + 1 \right) a_k \right) x^{k+r} \end{aligned}$$

Setting the total coefficient of each power of x equal to zero we obtain

$$\begin{aligned} r^2 &= 0 \\ a_{k+1} &= \frac{a_k}{k+r+1} \end{aligned}$$

The indicial equation $r^2 = 0$ implies $r = 0$, and so the recursion relations become

$$a_{k+1} = \frac{a_k}{k+1} \quad .$$

Hence

$$\begin{aligned}
a_1 &= \frac{a_0}{1} = a_0 \\
a_2 &= \frac{a_1}{2} = \frac{1}{2}a_0 \\
a_3 &= \frac{a_2}{3} = \frac{1}{(3)(2)}a_0 \\
a_4 &= \frac{a_3}{4} = \frac{1}{(4)(3)(2)}a_0 \\
&\vdots \\
a_n &= \frac{1}{n!}a_0
\end{aligned}$$

Thus,

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^{n+0} \\
&= \sum_{n=0}^{\infty} a_0 \frac{x^n}{n!} \\
&= a_0 e^x
\end{aligned}$$

$\square$