Math 2233
Homework Set 6

1. Solve the following Euler-type equations.

(a) $x^2y'' + xy' + y = 0$

- Since the differential equation is of Euler-type we expect solutions of the fomr $y(x) = x^r$. Substituting $y(x) = x^r$ into the differential equation we obtain

$$0 =^? x^2\left(r(r-1)x^{r-2}\right) + x\left(rx^{r-1}\right) + x^r = (r(r-1) + r + 1)\,x^r = \left(r^2+1\right)x^r$$

The indicial equation is thus

$$r^2 + 1 = 0$$

This equation has two complex roots $r = \pm i$. We thus have the following two linearly independent real-valued solutions

$$\begin{aligned} y_1(x) &= \cos(\ln|x|) \\ y_2(x) &= \sin(\ln|x|) \end{aligned}$$

and the general solution is

$$y(x) = c_1\cos(\ln|x|) + c_2\sin(\ln|x|)$$

(b) $x^2y'' - xy' + 2y = 0$

- Substituting $y(x) = x^r$ into this equation yields

$$0 = r(r-1)x^r - rx^r + 2x^r = \left(r^2 - 2r + 2\right)x^r$$

So we must have

$$r^2 - 2r + 2 = 0$$

or, after applying the quadratic formula,

$$r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

We thus have two complex roots of the indicial equation and thus the following two linearly independent real-valued solutions of the original Euler-type differential equation

$$\begin{aligned} y_1(x) &= x\cos(\ln|x|) \\ y_2(x) &= x\sin(\ln|x|) \end{aligned}$$

The general solution is thus

$$y(x) = c_1x\cos(\ln|x|) + c_2x\sin(\ln|x|)$$

(c) $4x^2y'' - 4xy' + 3y = 0$, $y(1) = 0$, $y'(1) = 1$

- Substituting $y(x) = x^r$ into this equation yields

$$0 = 4r(r-1)x^r - 4rx^r + 3x^r = \left(4r^2 - 8r + 3\right)x^r$$

So we must have

$$4r^2 - 8r + 3 = 0$$

or, after applying the quadratic formula,

$$r = \frac{8 \pm \sqrt{64-48}}{8} = \frac{8 \pm -\sqrt{16}}{8} = \frac{8 \pm 4}{8} = 1 \pm \frac{1}{2}$$

We thus have two real roots of the indicial equation; $r = \frac{3}{2}, \frac{1}{2}$. The corresponding the linearly independent solutions of the original Euler-type differential equation are

$$\begin{aligned} y_1(x) &= x^{3/2} \\ y_2(x) &= x^{1/2} \end{aligned}$$

The general solution is thus

$$y(x) = c_1 x^{3/2} + c_2 x^{1/2}$$

(d) $x^2y'' - 3xy' + 3y = 0$

- Substituting $y(x) = x^r$ into this equation yields

$$0 = r(r-1)x^r - 3rx^r + 3x^r = (r - 4r + 3)\, x^r$$

So we must have

$$0 = r^2 - 4r + 3 = (r-1)(r-3)$$

We thus have two real roots of the indicial equation; $r = 1, 3$. The corresponding linearly independent real-valued solutions of the original Euler-type differential equation are

$$\begin{aligned} y_1(x) &= x \\ y_2(x) &= x^3 \end{aligned}$$

The general solution is thus

$$y(x) = c_1 x + c_2 x^3$$

(e) $x^2y'' + 5xy' + 5y = 0$

- Substituting $y(x) = x^r$ into this equation yields

$$0 = r(r-1)x^r + 5rx^r + 5x^r = (r + 4r + 5)\, x^r$$

So we must have

$$0 = r^2 + 4r + 5$$

Applying the Quadratic Formula we find

$$r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm \frac{\sqrt{-4}}{2} = -2 \pm i$$

We thus have a pair of complex roots of the indicial equation; $r = -2 \pm i$. The corresponding linearly independent real-valued solutions of the original Euler-type differential equation are

$$\begin{aligned} y_1(x) &= x^{-2}\cos(\ln|x|) \\ y_2(x) &= x^{-2}\sin(\ln|x|) \end{aligned}$$

The general solution is thus

$$y(x) = c_1 x^{-2}\cos(\ln|x|) + c_2 x^{-2}\sin(\ln|x|)$$

2. Given that $y_1(x) = e^x$ and $y_2(x) = x$ are solutions of

$$(1-x)y'' + xy' - y = 0$$

find the general solution to

$$(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}$$

by the method of Variation of Parameters.

- In this problem we have

$$\begin{aligned} g(x) &= \frac{2(x-1)^2e^{-x}}{1-x} = 2(1-x)e^{-x} \\ W\left[y_1, y_2\right](x) &= e^x (x)' - (e^x)' x = e^x(1) - (e^x)\, x = (1-x)e^x \end{aligned}$$

and the Variation of Parameters formula yields

$$\begin{aligned} y_p(x) &= -y_1(x)\int^x \frac{y_2(s)g(s)}{W\left[y_1, y_2\right](s)}ds + y_2(x)\int^x \frac{y_1(s)g(s)}{W\left[y_1,y_2\right](s)}ds \\ &= -e^x \int^x \frac{s\left(2(1-s)e^{-s}\right)}{(1-s)e^s}ds + x\int^x \frac{e^s\left(2(1-s)e^{-s}\right)}{(1-s)e^s}ds \\ &= -2e^x \int^x se^{-2s}ds + 2x\int^x e^{-s}ds \\ &= -2e^x\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right) + 2x\left(-e^{-x}\right) \\ &= xe^{-x} + \frac{1}{2}e^{-x} - 2xe^{-x} \\ &= -xe^{-x} + \frac{1}{2}e^{-x} \end{aligned}$$

as a particular solution to the non-homogeneous differential equation. The general solution is thus

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) = -xe^{-x} + \frac{1}{2}e^{-x} + c_1e^x + c_2x$$

3. Find the general solution of each of the following equations by the method of Variation of Parameters. If initial conditions are given, find the solution satisfying that initial value problem.

(a) $y'' - 3y' + 2y = 10$

- The corresponding homogeneous equation is

$$y'' - 3y' + 2y = 0 \tag{1}$$

which is second order linear with constant coefficients. Its characteristic equation is

$$\lambda^2 - 3\lambda + 2 = 0$$

or

$$(\lambda-2)(\lambda-1) = 0$$

and so we have two real roots $\lambda = 2, 1$ and two linearly independent solutions of (**??**):

$$\begin{aligned} y_1(x) &= e^{2x} \\ y_2(x) &= e^x \end{aligned}$$

We will now use the Variation of Parameters to determine a particular solution of the original non-homogeneous equation. First note that

$$\begin{aligned} g(x) &= 10 \\ W\left[y_1, y_2\right](x) &= \left(e^{2x}\right)\left(e^x\right) - \left(2e^{2x}\right)\left(e^x\right) = -e^{3x} \end{aligned}$$

and so

$$\begin{aligned}
y_p(x) &= -y_1(x)\int^x \frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds + y_2(x)\int^x \frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds \\
&= -e^{2x}\int^x \frac{e^s(10)}{-e^{3s}}ds + e^x\int^x \frac{e^{2s}(10)}{-e^{3s}}ds \\
&= +10e^{2x}\int^x e^{-2s}ds - 10e^x\int^x e^{-s}ds \\
&= 10e^{2x}\left(-\frac{1}{2}e^{-2x}\right) - 10e^x\left(-e^{-x}\right) \\
&= -5+10 \\
&= 5
\end{aligned}$$

Thus the general solution is

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) = 5 + c_1e^{2x} + c_2e^x$$

(b) $y'' + y = \sin(x)$, $y(0) = 1$, $y'(0) = 2$

- The corresponding homogeneous equation is

(2)
$$y'' + y = 0$$

which is second order linear with constant coefficients. Its characteristic equation is

$$\lambda^2 + 1 = 0$$

or

$$(\lambda + i)(\lambda - i) = 0$$

and so we have two complex roots $\lambda = \pm 2i$ and two linearly independent solutions of (**??**):

$$\begin{aligned}
y_1(x) &= \cos(x) \\
y_2(x) &= \sin(x)
\end{aligned}$$

We will now use the Variation of Parameters to determine a particular solution of the original non-homogeneous equation. First note that

$$\begin{aligned}
g(x) &= \sin(x) \\
W[y_1,y_2](x) &= (\cos(x))(\cos(x)) - (\sin(x))(\sin(x)) = \cos^2(x) + \sin^2(x) = 1
\end{aligned}$$

and so

$$\begin{aligned}
y_p(x) &= -y_1(x)\int^x \frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds + y_2(x)\int^x \frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds \\
&= -\cos(x)\int^x \frac{\sin(s)(\sin(s))}{1}ds + \sin(x)\int^x \frac{\cos(s)(\sin(s))}{1}ds \\
&= -\cos(x)\int^x \sin^2(s)ds + \sin(x)\int^x \cos(s)\sin(s)ds \\
&= -\cos(x)\left(-\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}x\right) + \sin(x)\left(-\frac{1}{2}\cos^2(x)\right) \\
&= -\frac{x}{2}\cos(x)
\end{aligned}$$

:The general solution is thus

$$y(x) = -\frac{1}{2}x\cos(x) + c_1\cos(x) + c_2\sin(x)$$

We now impose the initial conditions to fix c_1 and c_2

$$\begin{aligned}
1 &= y(0) = -\frac{0}{2}\cos(0) + c_1\cos(0) + c_2\sin(0) \\
&= c_1
\end{aligned}$$

$$\begin{aligned}
2 &= y'(0) = -\frac{1}{2}\cos(x) + \frac{1}{2}x\sin(x) - c_1\sin(x) + c_2\cos(x)\Big|_{x=0} \\
&= -\frac{1}{2} + c_2 \\
\Rightarrow &\quad c_2 = \frac{5}{2}
\end{aligned}$$

Thus, $c_1 = 1$, $c_2 = 2$ and the solution to the initial value problem is

$$y(x) = -\frac{x}{2}\cos(x) + \cos(x) + \frac{5}{2}\sin(x)$$

(c) $y'' - 7y' + 10y = 100x$

- The corresponding homogeneous equation

(3) $$y'' - 7y' + 10y = 0$$

is second order linear with constant coefficients and has

$$\lambda^2 - 7\lambda + 10 = 0$$

as its characteristic equation. Since $\lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5)$ it's clear that we have two real roots $\lambda = 2, 5$ and two linearly independent solutions of (**??**)

$$\begin{aligned}
y_1(x) &= e^{2x} \\
y_2(x) &= e^{5x}
\end{aligned}$$

The Wronskian for these two solutions is

$$W[y_1, y_2](x) = e^{2x}\left(5e^{5x}\right) - \left(2e^{2x}\right)e^{5x} = 3e^{7x}$$

Takiing $g(x) = 100x$ we can now plug into the Variation of Paraemeters formula

$$\begin{aligned}
y_p(x) &= -y_1(x)\int^x \frac{y_2(s)g(s)}{W[y_1, y_2](s)}ds + y_2(x)\int^x \frac{y_1(s)g(s)}{W[y_1, y_2](s)}ds \\
&= -e^{5x}\int^x \frac{e^{2s}(100s)}{3e^{7s}}ds + e^{2x}\int^x \frac{e^{5s}(100s)}{3e^{7s}}ds \\
&= -\frac{100}{3}e^{5x}\int^x se^{-5s}ds + \frac{100}{3}e^{2x}\int^x se^{-2s}ds \\
&= -\frac{100}{3}e^{5x}\left(-\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x}\right) + \frac{100}{3}e^{2x}\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right) \\
&= \frac{20}{3}x + \frac{4}{3} - \frac{50}{3}x - \frac{25}{3} \\
&= -10x - 7
\end{aligned}$$

The general solution is thus

$$y(x) = -10x - 7 + c_1e^{2x} + c_2e^{5x}$$

(d) $y'' + 4y = \sec(2x)$

- The corresponding homogeneous equation

(4) $$y'' + 4y = 0$$

is second order linear with constant coefficients and has

$$\lambda^2 + 4 = 0$$

as its characteristic equation. Since $\lambda^2+4=(\lambda-2i)(\lambda+2i)$ it's clear that we have two complex roots $\lambda=\pm 2i$ and two linearly independent solutions of (**??**)

$$\begin{aligned} y_1(x) &= \cos(2x) \\ y_2(x) &= \sin(2x) \end{aligned}$$

Noting that

$$W[y_1,y_2](x)=\cos(2x)(2\cos(2x))-(-2\sin(2x))\sin(2x)=2\left(\cos^2(2x)+\sin^2(2x)\right)=2$$

and

$$g(x)=\sec(2x)=\frac{1}{\cos(2x)}$$

we can now plug into the Variation of Parameters formula

$$\begin{aligned} y_p(x) &= -y_1(x)\int^x \frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds+y_2(x)\int^x \frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds \\ &= -\cos(2x)\int^x \frac{\sin(2s)}{2\cos(2s)}ds+\sin(2x)\int^x \frac{\cos(2s)}{2\cos(2s)}ds \\ &= -\cos(2x)\left(-\frac{1}{4}\ln(\cos 2x)\right)+\sin(2x)\left(\frac{1}{2}x\right) \\ &= \frac{1}{4}\cos(2x)\ln(\cos(2x))+\frac{1}{2}x\sin(2x) \end{aligned}$$

The general solution of the original inhomogeneous differential equation is thus

$$y(x)=\frac{1}{4}\cos(2x)\ln(\cos(2x))+\frac{1}{2}x\sin(2x)+c_1\cos(2x)+c_2\sin(2x)$$

4. Use the method of Variation of Parameters to solve the following non-homogeneous Euler-type equation.

$$x^2y''-5xy'+9y=x^3$$

- First we solve the corresponding homogeneous equation

$$x^2y''-5xy'+9y$$

Substituting $y(x)=x^r$ into this equation yields

$$0=r(r-1)x^r-5rx^r+9x^r=(r-6r+9)x^r$$

So we must have

$$0=r^2-6r+9=(r-3)^2$$

We thus have a single real root of the indicial equation; $r=3$. The corresponding linearly independent real-valued solutions of the original Euler-type differential equation are

$$\begin{aligned} y_1(x) &= x^3 \\ y_2(x) &= x^3\ln|x| \end{aligned}$$

The Wronskian of $y_1(x)$ and $y_2(x)$ is

$$W[y_1,y_2](x)=x^3\left(3x^2\ln|x|+x^3\left(\frac{1}{x}\right)-\left(3x^2\right)x^3\ln|x|\right)=3x^5\ln|x|+x^5-3x^5\ln|x|=x^5$$

To identify the function $g(x)$ in the Variation of Parameters formula we must first cast the original non-homogeneous equation into standard form

$$y''-\frac{5}{x}y'+\frac{9}{x^2}y=x.$$

Hence, $g(x) = x$. Thus we have

$$\begin{aligned}
y_p(x) &= -y_1(x)\int^x \frac{y_2(s)g(s)}{W[y_1,y_2](s)}ds + y_2(x)\int^x \frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds \\
&= -x^3\int^x \frac{\left(s^3\ln|s|\right)s}{s^5}ds + x^3\ln|x|\int^x \frac{\left(s^3\right)s}{s^5}ds \\
&= -x^3\int^x \ln|s|\left(s^{-1}ds\right) + x^3\ln|x|\int^x \frac{1}{s}ds \\
&= -x^3\int^{u=\ln|x|} u\,du + x^3\ln|x|\left(\ln|x|\right) \\
&= -x^3\left(\frac{1}{2}\left(\ln|x|\right)^2\right) + x^3\left(\ln|x|\right)^2 \\
&= \frac{1}{2}x^3\left(\ln|x|\right)^2
\end{aligned}$$

Thus the general solution is

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) = \frac{1}{2}x^3\left(\ln|x|\right)^2 + c_1x^3 + c_2x^3\ln|x|$$

5. Find the general solution to the following differential equations. If initial conditions are specified, also determine the solution satisfying those initial conditions.

(a) $y^{(4)} + 2y'' + y = 0$

- The characteristic equation is

$$0 = \lambda^4 + 2\lambda^2 + y = (\lambda^2+1)^2 = \left((\lambda-i)(\lambda+i)\right)^2 = (\lambda-i)^2(\lambda+i)^2$$

We thus have two complex roots, $\lambda = +i, -i$ each with multiplicity two. The corresponding linearly independent solutions are

$$\begin{aligned}
y_1(x) &= \cos(x) \\
y_2(x) &= x\cos(x) \\
y_3(x) &= \sin(x) \\
y_4(x) &= x\sin(x)
\end{aligned}$$

and the general solution is

$$y(x) = c_1\cos(x) + c_2x\cos(x) + c_3\sin(x) + c_4x\sin(x)$$

(b) $y''' - y'' - y' + y = 0$

- The characteristic equation is

$$0 = \lambda^3 - \lambda^2 - \lambda + 1$$

Note that the right hand side vanishes when $\lambda = 1$; therefore $(\lambda-1)$ must be a factor of $\lambda^3 - \lambda^2 - \lambda + 1$. Indeed,

$$(\lambda-1)\,\overline{|\,\lambda^3 - \lambda^2 - \lambda + 1} = \lambda^2 - 1 = (x-1)(x+1)$$

So the characteristic polynomial factors as

$$\lambda^3 - \lambda^2 - \lambda + 1 = (\lambda-1)^2(\lambda+1)$$

Thus we have a double root at $\lambda = 1$ and a single root at $\lambda = -1$. The corresponding linearly independent solutions are

$$\begin{aligned}
y_1(x) &= e^x \\
y_2(x) &= xe^x \\
y_3(x) &= e^{-x}
\end{aligned}$$

and the general solution is

$$y(x) = c_1 e^x + c_2 x e^x + c_3 e^{-x}$$

(c) $y''' - 3y'' + 3y' - y = 0$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$

- The characteristic equation is

$$0 = \lambda^3 - 3\lambda^2 + 3\lambda - 1$$

Again $\lambda = 1$ is an obvious solution and so $(\lambda - 1)$ is a factor of $\lambda^3 - 3\lambda^2 + 3\lambda - 1$. To find the remaining factors we employ polynomial division and find

$$(\lambda - 1)\,|\,\overline{\lambda^3 - 3\lambda^2 + 3\lambda - 1} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

and so

$$0 = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)(\lambda - 1)^2 = (\lambda - 1)^3$$

We thus have a triple root $\lambda = 1$. The corresponding linearly independent solutions are

$$\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= xe^x \\ y_3(x) &= x^2 e^x \end{aligned}$$

and the general solution is

$$y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

We shall now impose the initial condtions to fix the arbitrary constants c_1, c_2, and c_3.

$$\begin{aligned} 1 &= y(0) = c_1 e^0 + c_2(0)e^0 + c_3(0)^2 e^0 = c_1 \\ 2 &= y'(0) = c_1 e^0 + c_2\left(e^0 + (0)e^0\right) + c_3\left(2(0)e^0 + (0)^2 e^0\right) = c_1 + c_2 \\ 3 &= y''(0) = c_1 e^0 + c_2\left(e^0 + e^0 + (0)e^0\right) + c_3 2e^o + 2(0)e^0 + 2(0)e^0 + (0)^2 e^0 = c_1 + 2c_2 + 2c_3 \end{aligned}$$

and so we have

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 2 - c_1 = 2 - 1 = 1 \\ c_3 &= \frac{1}{2}(3 - c_1 - 2c_2) = \frac{1}{2}(3 - 1 - 2) = 0 \end{aligned}$$

The solution to the initial value problem is thus

$$y(x) = e^x + xe^x$$

(d) $y''' + 5y'' - y' - 5y = 0$

- The characteristic equation is

$$0 = \lambda^3 + 5\lambda^2 - \lambda + 5$$

Again we are lucky enough to spot the solution $\lambda = 1$ and so we can identify the other roots by factoring the right hand side of

$$(\lambda - 1)\,|\,\overline{\lambda^3 + 5\lambda^2 - \lambda - 5} = \lambda^2 + 6\lambda + 5$$

Obviously, $\lambda^2 + 6\lambda + 5 = (\lambda + 5)(\lambda + 1)$, and so the right hand side of the characteristic equation factors as

$$0 = (\lambda - 1)(\lambda + 5)(\lambda + 1)$$

We thus have three distinct roots $\lambda = 1, -5, -1$. The corresponding linearly independent solutions are

$$\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= e^{-5x} \\ y_3(x) &= e^{-x} \end{aligned}$$

and the general solution is

$$y(x) = c_1 e^x + c_2 e^{-5x} + c_3 e^{-x}$$

(e) $y^{(4)} - 9y'' = 0$

The characteristic equation is

$$0 = \lambda^4 - 9\lambda^2 = \lambda^2(\lambda^2 - 9) = \lambda^2(\lambda - 3)(\lambda + 3) = (\lambda - 0)^2 (\lambda - 3)(+3)$$

We thus have a double root at $\lambda = 0$, and single roots at $\lambda = \pm 3$. The corresponding linearly independent solutions are

$$\begin{aligned} y_1(x) &= e^{0x} = 1 \\ y_2(x) &= xe^{0x} = x \\ y_3(x) &= e^{3x} \\ y_4(x) &= e^{-3x} \end{aligned}$$

and the general solution is

$$y(x) = c_1 + c_2 x + c_3 e^{3x} + c_4 e^{-3x}$$