Math 2233
Homework Set 5

1. Determine whether the given equation is linear or nonlinear. If it is linear, write it in standard form and state whether it is homogeneous or non-homogeneous.

(a) $xy'' + 2x^3y' + y = 0$

- The equation is linear. To put it in standard form we divide through by $x$ to get

$$y'' + 2x^2y' + \frac{1}{x} = 0.$$

  This is a homogeneous linear differential equation. $\square$

(b) $y'' + xy' + y^2 = 2x$

- This equation is non-linear due to the presence of the $y^2$ term. $\square$

(c) $3y'' + 2y' + y = x^5$

- This equation is linear. To put it in standard form we divide through by 3 to get

$$y'' + \frac{2}{3}y' + \frac{1}{3}y = \frac{1}{3}x^5.$$

  This equation is non-homogeneous, due to the presence of the $\frac{1}{3}x^5$ on the right hand side when it's written in standard form. $\square$

2. Verify that the two given functions are linearly independent solutions of the given homogeneous equation and then find the general solution.

(a) $y'' + 9y = 0$, $y_1(x) = \sin(3x)$, $y_2(x) = \cos(3x)$

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$$\begin{aligned}
y_1'' + 9y_1 &= (3\cos(3x))' + 9\sin(3x) \\
&= 3(-3\sin(3x)) + 9\sin(3x) \\
&= (-9+9)\sin(3x) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
y_2'' + 9y_2 &= (-3\sin(3x))' + 9\cos(3x) \\
&= -3(3\cos(3x)) + 9\cos(3x) \\
&= (-9+9)\cos(3x) \\
&= 0
\end{aligned}$$

  So both $y_1$ and $y_2$ are solutions. They are linearly independent since

$$\begin{aligned}
W[y_1, y_2](x) &\equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \\
&= \sin(3x)(-3\sin(3x)) - (3\cos(x))\cos(3x) \\
&= -3\left(\sin^2(x) + \cos^2(x)\right) \\
&= -3 \cdot 1 \\
&= -3 \\
&\neq 0
\end{aligned}$$

Since $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous linear equation $y'' + 9y = 0$, the general solution of this equation is

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sin(3x) + c_2 \cos(3x) \end{aligned}$$

□

(b) $y'' + 2y' - 15y = 0$, $y_1(x) = e^{3x}$, $y_2(x) = e^{-5x}$

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$$\begin{aligned} y_1'' + 2y_1' - 15y_1 &= (3)(3)e^{3x} + 2\left(3e^{3x}\right) - 15e^{3x} \\ &= (9 + 6 - 15)\, e^{3x} \\ &= 0 \end{aligned}$$

$$\begin{aligned} y_2'' + 2y_2' - 15y_2 &= (-5)(-5)e^{-5x} + 2\left(-5e^{-5x}\right) - 15e^{-5z} \\ &= (25 - 10 - 15)\, e^{-5x} \\ &= 0 \end{aligned}$$

So both $y_1(x)$ and $y_2(x)$ are solutions. They are also linearly independent since

$$\begin{aligned} W[y_1, y_2](x) &\equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= e^{3x}\left(-5e^{-5x}\right) - \left(3e^{3x}\right)e^{-5x} \\ &= (-5 - 3)\, e^{-2x} \\ &= -8e^{-3x} \\ &\neq 0 \end{aligned}$$

Since $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous linear equation $y'' + 2y' - 15y = 0$, the general solution of this equation is

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^{3x} + c_2 e^{-5x} \end{aligned}$$

□

(c) $y'' + 4y' + 4y = 0$, $y_1(x) = e^{-2x}$, $y_2(x) = xe^{-2x}$

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$$\begin{aligned} y_1'' + 4y_1' - 4y_1 &= (-2)(-2)e^{-2z} + 4\left(-2e^{-2x}\right) + 4e^{-2z} \\ &= (4 - 8 + 4)\, e^{-2x} \\ &= 0 \end{aligned}$$

$$\begin{aligned} y_2'' + 4y_2' + 4y_2 &= \left(e^{-2x} - 2xe^{-2x}\right)' + 4\left(e^{-2x} - 2xe^{-2x}\right) + 4xe^{-2x} \\ &= \left(-2e^{-2x} - 2e^{-2x} + 4xe^{-2x}\right) + 4\left(e^{-2x} - 2xe^{-2x}\right) + 4xe^{-2x} \\ &= (-2 - 2 + 4)\, e^{-2x} + (4 - 8 + 4)\, xe^{-2x} \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

So both $y_1(x)$ and $y_2(x)$ are solutions. They are also linearly independent since

$$\begin{aligned} W[y_1,y_2](x) &\equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= e^{-2x}\left(e^{-2x} - 2xe^{-2x}\right) - \left(-2e^{-2x}\right)xe^{-2x} \\ &= e^{-4x} - 2xe^{-4x} + 2xe^{-2x} \\ &= e^{-4x} \\ &\neq 0 \end{aligned}$$

Since $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous linear equation $y'' + 4y' + 4y = 0$, the general solution of this equation is

$$\begin{aligned} y(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1e^{-2x} + c_2xe^{-2x} \end{aligned}$$

□

3. Given that $y_1(x) = e^{3x}$ is one solution of $y'' - 5y' + 6y = 0$, find a second linearly independent solution and then write down the general solution.

- To find a second linearly independent solution we apply the Reduction of Order formula

$$y_2(x) = y_1(x)\int^x \frac{1}{[y_1(s)]^2}\exp\left[-\int^s p(t)dt\right]ds$$

For the case at hand, we have $y_1(x) = e^{3x}$ and $p(x) = -5$, so

$$\begin{aligned} y_2(x) &= e^{3x}\int^x \frac{1}{(e^{3s})^2}\exp\left[-\int^s(-5)dt\right]ds \\ &= e^{3x}\int^x e^{-6s}\exp[5s]\,ds \\ &= e^{3x}\int^x e^{-6s}e^{5s}ds \\ &= e^{3x}\int^x e^{-s}ds \\ &= e^{3x}\left(-e^{-x}\right) \\ &= -e^{2x} \\ &\sim e^{2x} \end{aligned}$$

In the last step we dropped the minus sign simply because if $-e^{2x}$ is a solution so is $e^{2x}$ (because of the Superposition Principle) , and the latter expression for $y_2(x)$ is a tad bit simpler.

The general solution is a linear combination of $y_1(x)$ and $y_2(x)$ :

$$y(x) = c_1e^{3x} + c_2e^{2x}.$$

4. Given that $y_1(x) = e^{2x}$ is one solution of $y'' - 4y = 0$, find a second linearly independent solution and then write down the general solution.

- We'll again apply the Reduction of Order formula to get a second linearly independent solution. For this problem we have $y_1(x) = e^{2x}$ and $p(x) = 0$. So

$$\begin{aligned} y_2(x) &= e^{2x} \int^x \frac{1}{(e^{2s})^2} \exp\left[-\int^s 0 \cdot dt\right] ds \\ &= e^{2x} \int^x e^{-4s} e^0 ds \\ &= e^{2x} \int^x e^{-4s} ds \\ &= e^{2x} \left(-\frac{1}{4} e^{-4x}\right) \\ &= -\frac{1}{4} e^{-2x} \\ &\sim e^{-2x} \end{aligned}$$

Again we have thrown away a constant factor to simplify the expression of $y_2(x)$. The general solution is thus

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}.$$

5. Given that $y_1(x) = x$ is one solution of $y'' - 2xy' + 2y = 0$, find a second linearly independent solution and then write down the general solution.

- We'll again apply the Reduction of Order formula to get a second linearly independent solution. For this problem we have $y_1(x) = x$ and $p(x) = -2x$. So

$$\begin{aligned} y_2(x) &= x \int^x \frac{1}{s^2} \exp\left[\int^s 2t dt\right] ds \\ &= x \int^x \frac{1}{s^2} \exp\left[s^2\right] ds \\ &= x \int^x \frac{e^{s^2}}{s^2} ds \end{aligned}$$

Unfortunately, we can not actually carry out the final integration to get a simple formula of $y_2(x)$. Nevertheless, the integral can at least always be evaluated numerically, and we can write the following formula for the general solution of the original differential equation

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x + c_2 x \int^x \frac{e^{s^2}}{s^2} ds\,.$$

6. Given that $y_1(x) = x\sin(x)$ is one solution of $x^2 y'' - 2xy' + (x^2 + 2)y = 0$, find a second linearly independent solution and then write down the general solution.

- We'll again apply the Reduction of Order formula to get a second linearly independent solution. This time we have $y_1(x) = x\sin(x)$ and $p(x) = -\frac{2}{x}$ (to identify $p(x)$ we first put the differential equation in standard form). So

$$\begin{aligned}
y_2(x) &= x\sin(x)\int^x \frac{1}{(s\sin(s))^2}\exp\left[-\int^s -\frac{2}{t}dt\right]ds \\
&= x\sin(x)\int^x \frac{1}{(s\sin(s))^2}\exp\left[2\ln|s\}\right]ds \\
&= x\sin(x)\int^x \frac{1}{(s\sin(s))^2}s^2 ds \\
&= x\sin(x)\int^x \frac{1}{\sin^2(s)}ds \\
&= x\sin(x)\int^x \csc^2(s)ds \\
&= x\sin(x)\left(-\cot(x)\right) \\
&= x\sin(s)\left(-\frac{\cos(x)}{\sin(x)}\right) \\
&= -x\cos(x) \\
&\sim x\cos(x)
\end{aligned}$$

where in the last step we dropped the factor of -1 to simplify the expression for $y_2(x)$.

The general solution is thus

$$y(x) = c_1 x\sin(x) + c_2 x\cos(x)\,.$$

7. Find the general solution of the following differential equations

(a) $y'' - 5y = 0$.

- This is a second order linear equation with constant coefficients and so we look for solutions of the form $y(x) = e^{\lambda x}$. Plugging $y(x) = e^{\lambda x}$ into the differential equation yields

$$\lambda^2 e^{\lambda x} - 5e^{\lambda x} = 0$$

or

$$\left(\lambda^2 - 5\right)e^{\lambda x} = 0$$

Thus the characteristic equation for this differential equation is

$$\lambda^2 - 5 = 0$$

or

$$\lambda^2 = 5$$

which obviously has as solutions

$$\lambda = \pm\sqrt{5}$$

So both $y_1(x) = e^{\sqrt{5}x}$ and $y_2(x) = e^{-\sqrt{5}x}$ are solutions of the differential equation. Moreover, they are linearly independent since

$$W\left[y_1, y_2\right](x) = e^{\sqrt{5}x}\left(-\sqrt{5}e^{-\sqrt{5}x}\right) - \left(\sqrt{5}e^{\sqrt{5}x}\right)e^{-\sqrt{5}x} = -2\sqrt{5} \neq 0.$$

Therefore the general solution of the differential equation is

$$y(x) = c_1 e^{\sqrt{5}x} + c_2 e^{-\sqrt{5}x}\,.$$

(b) $y'' - 3y + 2y = 0$

- The characteristic equation for this second order linear equation with constant coefficients is

$$\lambda^2 - 3\lambda + 2 = 0$$

or

$$(\lambda - 1)(\lambda - 2) = 0$$

Thus, we have $\lambda = 1, 2$ as solutions. To each of these roots of the characterisitic equation we have a corresponding solution of the original differential equation; namely, $y_1(x) = e^x$ and $y_2(x) = e^{2x}$. These two solutions are linearly independent and so the general solution is

$$y(x) = c_1 e^x + c_2 e^{2x} \, .$$

(c) $y'' - y' - 20y = 0$

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 - \lambda - 20 = 0$$

or

$$(\lambda - 5)(\lambda + 4) = 0.$$

Thus, $\lambda = 5, -4$ and we have two linearly independent solutions $y_1(x) = e^{5x}$ and $y_2(x) = e^{-4x}$. The general solution is thus

$$y(x) = c_1 e^{5x} + c_2 e^{-4x} \, .$$

(d) $y'' - 13y' + 42y = 0$

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 - 13\lambda + 42 = 0$$

To solve the characteristic equation we apply the Quadratic Formula:

$$\begin{aligned} \lambda &= \frac{-(-13) \pm \sqrt{(-13)^2 - 4\,(42)}}{2} \\ &= \frac{13 \pm \sqrt{169 - 168}}{2} \\ &= \frac{13 \pm 1}{2} \\ &= 7, 6 \end{aligned}$$

Thus, $\lambda = 6, 7$ and we have two linearly independent solutions $y_1(x) = e^{6x}$ and $y_2(x) = e^{7x}$. The general solution is thus

$$y(x) = c_1 e^{6x} + c_2 e^{7x}$$

(e) $y'' + y' + 7y = 0$

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 + \lambda + 7 = 0$$

To solve the characteristic equation we apply the Quadratic Formula:

$$\begin{aligned} \lambda &= \frac{-1 \pm \sqrt{(1)^2 - (4)(7)}}{2} \\ &= \frac{-1 \pm \sqrt{1 - 28}}{2} \\ &= \frac{-1 \pm \sqrt{-27}}{2} \\ &= \frac{-1 \pm \sqrt{27}i}{2} \end{aligned}$$

Thus, we have a pair of complex roots

$$\lambda = -\frac{1}{2} + \frac{\sqrt{27}}{2}i\,,\ -\frac{1}{2} - \frac{\sqrt{27}}{2}i$$

and so we take

$$\begin{aligned} \alpha &= Re(\lambda) = -\frac{1}{2} \\ \beta &= \pm Im(\lambda) = \frac{\sqrt{27}}{2} \end{aligned}$$

Associated to the complex roots $\lambda = \alpha \pm i\beta$ are the following real-valued solutions of the original differential equation:

$$\begin{aligned} y_1(x) &= e^{\alpha x}\cos(\beta x) = e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{27}}{2}x\right) \\ y_2(x) &= e^{\alpha x}\sin(\beta x) = e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{27}}{2}x\right) \end{aligned}$$

and so the general solution is

$$y(x) = c_1 e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{27}}{2}x\right) + c_2 e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{27}}{2}x\right)\ .$$

(f) $y'' + 2y' + 5y = 0$

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 + 2\lambda + 5 = 0$$

To solve the characteristic equation we apply the Quadratic Formula:

$$\begin{aligned} \lambda &= \frac{-2 \pm \sqrt{(2)^2 - (4)(5)}}{2} \\ &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\ &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i \end{aligned}$$

Thus, we have a pair of complex roots

$$\lambda = -1 + 2i\,,\ -1 - 2i$$

so we take $\alpha = -1 = Re\,(\lambda)$ and $\beta = 2 = \pm Im(\lambda)$. Associated to this pair of complex roots are the following real-valued solutions of the original differential equation:

$$\begin{aligned} y_1(x) &= e^{\alpha x}\cos(\beta x) = e^{-x}\cos(2x) \\ y_2(x) &= e^{\alpha x}\sin(\beta x) = e^{-x}\sin(2x) \end{aligned}$$

and so the general solution is

$$y(x) = c_1 e^{-x}\cos(2x) + c_2 e^{-x}\sin(2x)\ .$$

8. Solve the following initial value problems.

(a) $y'' - 9y = 0$, $y(0) = 1$, $y'(0) = 2$.

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 - 9 = 0$$

or

$$(\lambda - 3)(\lambda + 3) = 0$$

so $\lambda = 3, -3$. The general solution is thus

$$y(x) = c_1 e^{3x} + c_2 e^{-3x} .$$

We now impose the initial conditions to fix the constants $c_1$ and $c_2$.

$$\begin{aligned} 1 &= y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 \\ 2 &= y'(0) = 3c_1 e^{3x} - 3c_2 e^{-3x}\big|_{x=0} = 3c_1 - 3c_2 \end{aligned}$$

Thus we have two equations and two unknowns

$$\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 - 3c_2 &= 2 \end{aligned}$$

Adding 3 times the first equation to the second equation yields

$$6c_1 + 0 = 5$$

so $c_1 = \frac{5}{6}$. But then the equation $c_1 + c_2 = 1$ implies that $c_2 = \frac{1}{6}$. Thus, the solution to the initial value problem is

$$y(x) = \frac{5}{6} e^{3x} + \frac{1}{6} e^{-3x} .$$

(b) $y'' - 2y' + y = 0$, $y(0) = 2$, $y'(0) = 1$.

- he characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 - 2\lambda + 1 = 0$$

or

$$(\lambda - 1)^2 = 0$$

so $\lambda = 1$. So $\mathrm{y}_1(x) = e^x$ is one solution of the differential equation, and (because we are in the case where there is only one distinct root for the characteristic equation) $y_2(x) = xy_1(x) = xe^x$ is a second linearly independent solution. The general solution is thus

$$y(x) = c_1 e^x + c_2 x e^x .$$

We now impose the initial conditions to fix $c_1$ and $c_2$:

$$\begin{aligned} 2 &= y(0) = c_1 e^0 + c_2 (0) e^0 = c_1 \\ 1 &= y'(0) = c_1 e^x + c_2 (e^x + xe^x)\big|_{x=0} = c_1 + c_2 \end{aligned}$$

Thus, we must have

$$\begin{aligned} c_1 &= 2 \\ c_1 + c_2 &= 1 \end{aligned}$$

Obviously, we must have $c_1 = 2$ and $c_2 = -1$. Thus, the solution to the initial value problem is

$$y(x) = 2e^x - xe^x .$$

(c) $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = -1$

- The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 + 2\lambda + 2 = 0$$

The Quadratic Formula thus implies

$$\begin{aligned} \lambda &= \frac{-2 \pm \sqrt{(2)^2 - (4)(2)}}{2} \\ &= \frac{-2 \pm \sqrt{-4}}{2} \\ &= \frac{-2 \pm 2i}{2} \\ &= -1 \pm i \end{aligned}$$

We thus have a pair of complex roots $\lambda = \alpha \pm i\beta$ with $\alpha = -1$ and $\beta = 1$. The general solution is thus

$$y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x) = c_1 e^{x} \cos(x) + c_2 e^{x} \sin(x)\,.$$

We now impose the initial conditions to fix $c_1$ and $c_2$:

$$\begin{aligned} 1 &= y(0) = c_1 e^0 \cos(0) + c_2 e^0 \sin(0) = c_1(1)(1) + c_2(1)(0) = c_1 \\ -1 &= y'(0) = c_1 \left(e^x \cos(x) - e^x \sin(x)\right) + c_2 \left(e^x \sin(x) + e^x \cos(x)\right)\big|_{x=0} = c_1(1-0) + c_2(0+1) = c_1 + c_2 \end{aligned}$$

We thus have

$$\begin{aligned} c_1 &= 1 \\ c_1 + c_2 &= -1 \end{aligned}$$

which implies $c_1 = 1$ and $c_2 = -2$. Thus, the solution to the initial value problem is

$$y(x) = e^x \cos(x) - 2e^x \sin(x)\,.$$