Math 2233
Homework Set 4

1. Find an integrating factor for each of the following differential equations and obtain the general solution.

(a) $y + (y - x)y' = 0$

- Suppose $\mu(x, y)$ is an integrating factor for this equation. Then
$$\mu y + \mu(y - x)y' = 0$$
must be exact so
$$\frac{\partial}{\partial y}(\mu y) = \frac{\partial}{\partial x}(\mu y - \mu x)$$
or
$$\frac{\partial \mu}{\partial y} y + \mu = \frac{\partial \mu}{\partial x} y - \frac{\partial \mu}{\partial x} x - \mu$$
In order to simplify this equation we make the hypothesis that $\mu(x, y) = \mu(y)$. Then $\frac{\partial \mu}{\partial y} = \frac{d\mu}{dy}$ and $\frac{\partial \mu}{\partial x} = 0$; so we have
$$y \frac{d\mu}{dy} + \mu = -\mu$$
or
$$\frac{d\mu}{dy} + \frac{2}{y}\mu = 0.$$
This is a first order linear equation for μ. It is also separable, since we can rewrite it as
$$\frac{d\mu}{\mu} = -\frac{2dy}{y}$$
Integrating both sides yields
$$\ln|\mu| = -2\ln|y|$$
or
$$\mu = e^{-2\ln|y|} = e^{\ln\left|y^{-2}\right|} = y^{-2}.$$
Therefore y^{-2} should be our integrating factor. Multiplying the original differential equation by y^{-2} yields
$$\frac{1}{y} + \left(\frac{1}{y} - \frac{x}{y^2}\right) y' = 0$$
Now we have
$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{1}{y^2} \\
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y} - \frac{x}{y^2}\right) = -\frac{1}{y^2}
\end{aligned}$$
Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact. Therefore the differential equation is equivalent to a (family of) algebraic equation(s) of the form
$$\phi(x, y) = C$$
with
$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M = \frac{1}{y} \\
\frac{\partial \phi}{\partial y} &= N = \frac{1}{y} - \frac{x}{y^2}
\end{aligned}$$

Taking anti-partial derivatives of both these equations yields the following two expressions for ϕ;

$$\begin{aligned}\phi(x,y) &= \int \frac{\partial\phi}{\partial x}\partial x = \int \left(\frac{1}{y}\right)\partial x = \frac{x}{y} + H_1(y) \\ \phi(x,y) &= \int \frac{\partial\phi}{\partial y}\partial y = \int \left(\frac{1}{y} - \frac{x}{y^2}\right)\partial y = \ln|y| + \frac{x}{y} + H_2(x)\end{aligned}$$

Comparing these two expressions we see we must take $H_1(y) = \ln|y|$, $H_2(x) = 0$, and so $\phi(x,y) = \ln|y| + \frac{x}{y}$. Hence, an implicit solution to the original differential equation is

$$\ln|y| + \frac{x}{y} = C.$$

(b) $x^2 + y^2 + x + yy' = 0$

- If $\mu(x,y)$ is a integrating factor for this equation we must have

$$\frac{\partial}{\partial y}\left(\mu x^2 + \mu y^2 + \mu x\right) = \frac{\partial}{\partial x}(\mu y)$$

or

$$\frac{\partial\mu}{\partial y}x^2 + \frac{\partial\mu}{\partial y}y^2 + \mu(2y) + \frac{\partial\mu}{\partial y}x = \frac{\partial\mu}{\partial x}y$$

If we suppose $\mu(x,y)$ actually only depends on x then the above partial differential equation for μ simplifies to

$$2y\mu = y\frac{d\mu}{dx}$$

or

$$\frac{d\mu}{\mu} = 2dx$$

Integrating both sides yields

$$\ln|\mu| = 2x$$

or

$$\mu(x) = e^{2x}.$$

Hence, e^{2x} should be an integrating factor for the original differential equation. Hence,

$$x^2e^{2x} + y^2e^{2x} + xe^{2x} + ye^{2x}y' = 0$$

should be exact. Indeed,

$$\frac{\partial}{\partial y}\left(x^2e^{2x} + y^2e^{2x} + xe^{2x}\right) = 2ye^{2x} = \frac{\partial}{\partial x}\left(ye^{2x}\right)$$

so the equation is exact. Therefore we look for a function $\phi(x,y)$ such that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= x^2e^{2x} + y^2e^{2x} + xe^{2x} \\ \frac{\partial\phi}{\partial y} &= ye^{2x}\end{aligned}$$

Taking anti-partial derivatives of both these equations yields

$$\begin{aligned}\phi(x,y) &= \int \left(x^2e^{2x} + y^2e^{2x} + xe^{2x}\right)\partial x = \frac{1}{2}x^2e^{2x} + \frac{1}{2}y^2e^{2x} + H_1(y) \\ \phi(x,y) &= \int ye^{2x}\partial y = \frac{1}{2}y^2e^{2x} + H_2(x)\end{aligned}$$

Comparing these two expressions for $\phi(x,y)$ we see that we must take $H_1(y) = 0$, $H_2(x) = \frac{1}{2}x^2e^{2x}$, and so $\phi(x,y) = \frac{1}{2}x^2e^{2x} + \frac{1}{2}y^2e^{2x}$.. Thus, the general (implicit) solution of the original differential equation is

$$\frac{1}{2}x^2e^{2x} + \frac{1}{2}y^2e^{2x} = C$$

Solving for y yields

$$y(x) = \pm\sqrt{2Ce^{-2x} - x^2}$$

(c) $2y^2 + (2x + 3xy)y' = 0$

- If $\mu(x,y)$ is to be an integrating factor we must have

$$\frac{\partial}{\partial y}\left(2y^2\mu\right) = \frac{\partial}{\partial x}\left(2x\mu + 3xy\mu\right)$$

or

$$4y\mu + 2y^2\frac{\partial\mu}{\partial y} = 2\mu + 2x\frac{\partial\mu}{\partial x} + 3y\mu + 3xy\frac{\partial\mu}{\partial x}$$

This equation would simplify tremedously if we supposed that $\mu(x,y)$ actually depended only on y. Then we would have

$$4y\mu + 2y^2\frac{d\mu}{dy} = 2\mu + 3y\mu$$

or

$$2y^2\frac{d\mu}{dy} = (2-y)\,\mu$$

or

$$\frac{d\mu}{\mu} = \left(\frac{1}{y^2} - \frac{1}{2y}\right)dy$$

Integrating both sides yields

$$\ln|\mu| = -\frac{1}{y} - \frac{1}{2}\ln|y|$$

or

$$\mu = \exp\left(-\frac{1}{y} + \ln|y^{-\frac{1}{2}}|\right) = \frac{1}{y^{1/2}}e^{-\frac{1}{y}}.$$

So $\frac{1}{\sqrt{y}}e^{-\frac{1}{y}}$ should be an integrating factor for the original differential equation. Indeed, multiplying the original equation by $\frac{1}{\sqrt{y}}e^{-\frac{1}{y}}$ yields

$$2y^{3/2}e^{-\frac{1}{y}} + \left(2\frac{x}{y^{1/2}}e^{-\frac{1}{y}} + 3y^{1/2}xe^{-\frac{1}{y}}\right)y'$$

which is exact since

$$\frac{\partial}{\partial y}\left(2y^{3/2}e^{-\frac{1}{y}}\right) = 3y^{1/2}e^{-\frac{1}{y}} + 2y^{3/2}\left(+\frac{1}{y^2}e^{-\frac{1}{y}}\right) = 3y^{1/2}e^{-\frac{1}{y}} + 2y^{-1/2}e^{-\frac{1}{y}}$$

$$\frac{\partial}{\partial x}\left(2\frac{x}{y^{1/2}}e^{-\frac{1}{y}} + 3y^{1/2}xe^{-\frac{1}{y}}\right) = 2y^{-1/2}e^{-\frac{1}{y}} + 3y^{1/2}e^{-\frac{1}{y}}$$

and so we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.Now we need to find a $\phi(x,y)$ such that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= 2y^{3/2}e^{-\frac{1}{y}}\\ \frac{\partial\phi}{\partial y} &= 2\frac{x}{y^{1/2}}e^{-\frac{1}{y}} + 3y^{1/2}xe^{-\frac{1}{y}}\end{aligned}$$

Taking ant-partial derivatives of both of these equations yields

$$\begin{aligned}\phi(x,y) &= 2y^{3/2}e^{-\frac{1}{y}}x + H_1(y)\\ \phi(x,y) &= 2xy^{3/2}e^{-\frac{1}{y}} + H_2(x)\end{aligned}$$

Hence, we must have $H_1(y) = 0$, $H_2(x) = 0$, and $\phi(x,y) = 2y^{3/2}e^{-\frac{1}{y}}$. Thus, the general solution to the original differential equation is given implicitly by

$$2xy^{3/2}e^{-\frac{1}{y}} = C.$$

(d) $xy - x^2y' = 0$

- If $\mu(x,y)$ is to be an integrating factor we must have

$$\frac{\partial}{\partial y}(xy\mu) = \frac{\partial}{\partial x}\left(-x^2\mu\right)$$

or

$$x\mu + xy\frac{\partial \mu}{\partial y} = -2x\mu - x^2\frac{\partial \mu}{\partial x}.$$

This equation simplifies tremendously if we can assume μ depends only on x. In this case, the partial differential equation for μ becomes

$$x\mu + 0 = -2x\mu - x^2\frac{d\mu}{dx}$$

or

$$x^2\frac{d\mu}{dx} = -3x\mu$$

or

$$\frac{d\mu}{\mu} = -\frac{3}{x}dx$$

Integrating both sides yieldsx

$$\ln|\mu| = -3\ln|x|. = \ln\left|x^{-3}\right|$$

or $\mu = x^{-3}$ Multiplying the original differential equation by this function yields

$$x^{-2}y - x^{-1}y' = 0.$$

This equation is exact since

$$\begin{aligned}\frac{\partial}{\partial y}(x^{-2}y) &= x^{-2}\\ \frac{\partial}{\partial x}\left(-x^{-1}\right) &= x^{-2}\end{aligned}$$

so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Now we look for a $\phi(x,y)$ such that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= x^{-2}y\\ \frac{\partial \phi}{\partial y} &= -x^{-1}\end{aligned}$$

Taking anti-partial derivatives of each of these equations yields

$$\begin{aligned}\phi(x,y) &= \int x^{-2}y\partial x = -x^{-1}y + H_1(y)\\ \phi(x,y) &= \int -x^{-1}\partial y = -x^{-1}y + H_2(x).\end{aligned}$$

So we must have $H_1(y) = 0$, $H_2(x) = 0$, and so $\phi(x,y) = x^{-1}y = y/x$ The original differential equation is thus equivalent to an algebraic relation of the form

$$\frac{y}{x} = C$$

or

$$y = Cx.$$

2. Solve the following first order differential equations using the substitution $u = y/x$.

(a) $xy' - y = \sqrt{xy}$

- This equation is equivalent to

$$y' = \frac{1}{x}\left(\sqrt{xy} + y\right) = \sqrt{\frac{y}{x}} + \frac{y}{x} = F\left(\frac{y}{x}\right) \tag{1}$$

where

$$F(u) \equiv u^{1/2} + u.$$

Thus the equation is homogeneous of degree zero. If we define $u \equiv y/x$, we have $u' = y'/x - y/x^2$ and solving the latter equation for y' yields

$$y' = xu' + \frac{y}{x} = xu' + u.$$

If we now substitute $xu' + u$ for y' on the left hand side of (1) we obtain

$$xu' + u = y' = \sqrt{\frac{y}{x}} + \left(\frac{y}{x}\right) = u^{1/2} + u$$

or, cancelling the term u that appears on both sides,

$$xu' = u^{1/2}$$

or

$$u^{-1/2}du = \frac{dx}{x}.$$

Integrating both sides yields

$$2u^{1/2} = \ln|x| + C$$

or

$$2\sqrt{\frac{y}{x}} = \ln|x| + C.$$

Solving for y yields

$$y(x) = \frac{1}{4}x\left(\ln|x| + C\right)^2$$

(b) $y' = \dfrac{y^2 + xy}{x^2}$, $y(1) = 1$

- This equation is equivalent to

$$y' = \frac{\frac{1}{x^2}}{\frac{1}{x^2}}\left(\frac{y^2 + xy}{x^2}\right) = \frac{\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)}{1} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) = F\left(\frac{y}{x}\right)$$

where

$$F(u) \equiv u^2 + u.$$

Substituting $y = ux$, and $y' = xu' + u$, we obtain

$$xu' + u = y' = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right) = u^2 + u$$

or

$$xu' = u^2$$

or

$$\frac{du}{u^2} = \frac{dx}{x}.$$

Integrating both sides yields

$$-\frac{1}{u} = \ln|x| + C$$

or

$$-\frac{x}{y} = \ln|x| + C. \tag{2}$$

At this point it is convenient to employ the initial condition $y(1) = 1$ to fix the constant of integration C. Setting $x = 1$ and $y = 1$ in the equation above yields

$$-1 = \ln|1| + C = 0 + C = C.$$

So $C = 1$. Now we set $C = 1$ in (2) and solve for y to get

$$y(x) = \frac{-x}{\ln|x| - 1}.$$

(c) $3xyy' + x^2 + y^2 = 0$

- This equation is equivalent to

$$y' = \frac{-x^2 - y^2}{3xy} = -\frac{1}{3}\left(\frac{x}{y}\right) - \frac{1}{3}\left(\frac{y}{x}\right)$$

Making the substitutions $y = ux$, $y' = xu' + u$ yields

$$xu' + u = -\frac{1}{3}u^{-1} - \frac{1}{3}u = -\frac{1}{3}\left(\frac{1}{u} + u\right) = \frac{-1 - u^2}{3u}$$

or

$$xu' = \frac{-1 - u^2}{3u} - u = \frac{-1 - u^2 - 4u^3}{3u} = -\frac{1}{3}\left(1 + 4u^2\right)$$

or, after dividing the extreme sides both by $x\left(1 + 4u^2\right)$,

$$\frac{u}{1 + 4u^2}\frac{du}{dx} = -\frac{1}{3x}$$

Integrating both sides yields

$$\frac{1}{8}\ln\left(1 + 4u^2\right) = \int \frac{udu}{1 + 4u^2} = -\frac{1}{3}\ln|x| + C$$

or

$$\ln\left[1 + 4\left(\frac{y}{x}\right)^2\right] = C' + \ln|x^{-8/3}|$$

or

$$1 + 4\left(\frac{y}{x}\right)^2 = \exp\left[C' + \ln|x^{-8/3}|\right] = e^{C'}\exp\left[\ln|x^{-8/3}|\right] = C''x^{-8/3}$$

or, solving for y,

$$y(x) = \pm\frac{1}{2x}\sqrt{C''x^{-8/3} - 1}$$

3. Find a substitution that provides a solution to the following differential equations.

(a) $xy' + y = (xy)^3$

- In the hopes of simplifying the right hand side of the differential equation, we'll try $z = xy$. In this case, we'll have $y = z/x$ and $y' = z'/x - z/x^2$. Making the corresponding substitutions into the differential equation yields

$$x\left(\frac{1}{x}z' - \frac{z}{x^2}\right) + \frac{z}{x} = z^3$$

or

$$z' = z^3$$

or

$$\frac{dz}{z^3} = dx.$$

Integrating both sides yields

$$-\frac{1}{2}z^{-2} = x + C$$

or

$$z = \pm\frac{1}{\sqrt{(-2x - 2C)}}$$

Recalling that $z \equiv xy$, we obtain

$$yx = \pm \frac{1}{\sqrt{(-2x-2C)}}$$

or

$$y(x) = \frac{\pm 1}{x\sqrt{(-2x-2C)}}$$

(b) $(x+y)y' = (2x+2y) \; - 3$

- For this equation it makes sense to try a substitution of the form $z = x+y$. In this case, we would have $y' = z' - 1$ and making

  the corresponding substitutions into the original differential equation would yield

$$z(z'-1) = 2z - 3$$

or

$$z' = \frac{1}{z}(3z-3) = \frac{3z-3}{z}$$

or

$$\frac{zdz}{3z-3} = dx.$$

Integrating both sides yields

$$= \frac{1}{3}z + \frac{1}{3}\ln(3z-3) = \int \frac{zdz}{3z-3} = \int dx + C = x + C$$

or

$$\frac{1}{3}(x+y) + \frac{1}{3}\ln(3(x+y)-3) - x = C$$

(Since we won't be able to solve this equation explicitly for y, the best we can do is express the solution as an algebraic equation.)