Math 2233
Homework Set 3

1. Solve the following initial value problem.

$$y' - y = 2xe^{2x} \quad ; \quad y(1) = 0 \; .$$

- This is a first order linear ODE with $p(x) = -1$ and $g(x) = 2xe^{2x}$. So

$$\mu(x) = \exp\left[\int^x p(s)ds\right] = \exp\left[\int^x -ds\right] = \exp[-x] = e^{-x}$$

hence, the general solution of the ODE is

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^x \mu(s)g(s)ds + \frac{C}{\mu(x)} \\ &= \frac{1}{e^{-x}}\int^x e^{-s}\left(2se^{2s}\right)ds + \frac{C}{e^{-x}} \\ &= e^x\int^x 2se^s ds + Ce^x \\ &= e^x\left(2xe^x - 2e^x\right) + Ce^x \\ &= 2xe^{2x} - 2e^{2x} + Ce^x \end{aligned}$$

We now impose the initial condition $y(1) = 0$:

$$\begin{aligned} 0 = y(1) &= 2(1)e^2 - 2e^2 + Ce^1 \\ &= Ce \end{aligned}$$

Thus, $C = 0$ and so the solution to the initial value problem is

$$y(x) = 2xe^{2x} - 2e^{2x}.$$

2. Solve the following initial value problem.

$$y' + \frac{2}{x}y = \frac{\cos(x)}{x^2} \quad ; \quad y(\pi) = 0$$

- This is a first order linear ODE with $p(x) = \frac{2}{x}$ and $g(x) = \frac{\cos(x)}{x^2}$. Hence

$$\mu(x) = \exp\left[\int^x p(s)ds\right] = \exp\left[\int^x \frac{2}{s}ds\right] = \exp\left[2\ln|x|\right] = \exp\left[\ln|x^2|\right] = x^2$$

and so the general solution of the ODE is

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^x \mu(s)g(s)ds + \frac{C}{\mu(x)} \\ &= \frac{1}{x^2}\int^x s^2\left(\frac{\cos(s)}{s^2}\right)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\int^x \cos(s)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\sin(x) + \frac{C}{x^2} \end{aligned}$$

We now impose the initial condition to fix C.

$$0 = y(\pi) = \frac{1}{\pi^2}\sin(\pi) + \frac{C}{\pi^2} = 0 + \frac{C}{\pi^2} = \frac{C}{\pi^2}$$

So we must take $C = 0$. The solution to the initial value problem is thus

$$y(x) = \frac{\sin(x)}{x^2}.$$

3. Find the solution of the initial value problem below. State the interval in which the solution is valid.

$$xy' + 2y = x^2 - x + 1 \;\; ; \;\; y(1) = \frac{1}{2} \; .$$

- Dividing both sides by x we put the differential equation in standard form:

$$y' + \frac{2}{x}y = x - 1 + \frac{1}{x}$$

so $p(x) = \frac{2}{x}$ and $g(x) = x - 1 + \frac{1}{x}$. Note that since $p(x)$ and $g(x)$ are both undefined at $x = 0$, we might expect trouble for any solution we construct at the point $x = 0$. At any rate

$$\mu(x) = \exp\left[\int^x p(s)ds\right] = \exp\left[\int^x \frac{2}{s}ds\right] = \exp\left[2\ln|x|\right] = x^2$$

and so the general solution is

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^x \mu(s)g(s)ds + \frac{C}{\mu(x)} \\ &= \frac{1}{x^2}\int^x s^2\left(s - 1 + \frac{1}{s}\right)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\int\left(s^3 - s^2 + s\right)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\left(\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2\right) + \frac{C}{x^2} \\ &= \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{C}{x^2} \end{aligned}$$

Note that if $C \neq 0$ then a solution is undefined at $x = 0$. Now we plug into the initial condition

$$\frac{1}{2} = y(1) = \frac{1}{4}(1)^2 - \frac{1}{3}(1) + \frac{1}{2} + \frac{C}{(1)^2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C = \frac{5}{12} + C$$

so $C = \frac{1}{12}$.Thus the solution to the initial value problem is

$$y(x) = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{1}{12}x^{-2}$$

which is which is well-defined on any interval that excludes the point $x = 0$.

4. Find the solution of the initial value problem below. State the interval in which the solution is valid.

$$y' + y = \frac{1}{1+x^2} \quad , \quad y(0) = 0 \quad .$$

- The differential equation is in standard form and the coefficient functions $p(x) = 1$ and $g(x) = \frac{1}{1+x^2}$ are well-defined for all x so we can expect solutions to be well defined on any subinterval of the real line. Calculating $\mu(x)$ we get

$$\mu(x) = \exp\left[\int^x p(s)ds\right] = \exp\left[\int^x ds\right] = e^x$$

and so the general solution will be

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^x \mu(s)g(s)ds + \frac{C}{\mu(x)} \\ &= \frac{1}{e^x}\int^x \frac{e^s}{1+s^2}ds + \frac{C}{e^x} \\ &= e^{-x}\int^x \frac{e^s}{1+s^2}ds + \frac{C}{e^x} \end{aligned}$$

Unfortunately, there is no way to evaluate the integral

$$\int \frac{e^x}{1+x^2} dx$$

in closed form. To make further progress, we need to use the following formula for the solution of an initial value problem of the form $y' + p(x)y = g(x)$, $y(x_o) = y_o$.

$$y(x) = \frac{1}{\mu_o(x)} \int_{x_o}^{x} \mu_o(s) g(s) ds + \frac{y_o}{\mu_o(x)} \tag{1}$$

where

$$\mu_o(x) = \exp\left[\int_{x_o}^{x} p(s) ds\right] \tag{2}$$

Note the use of definite integrals in these formulas. In accordance with the initial condition $y(0) = 0$, we set $x_o = 0$ and $y_o = 0$; and

plug into the formulas (**??**) and (**??**):

$$\mu_o(x) = \exp\left[\int_0^x p(s) ds\right] = \exp\left[\int_0^x ds\right] = \exp\left[x - 0\right] = e^x$$

$$\begin{aligned} y(x) &= \frac{1}{\mu_o(x)} \int_{x_o}^{x} \mu_o(s) g(s) ds + \frac{y_o}{\mu_o(x)} \\ &= \frac{1}{e^x} \int_0^x \frac{e^s}{1+s^2} ds + \frac{0}{e^x} \\ &= \frac{1}{e^x} \int^x \frac{e^s}{1+s^2} ds \end{aligned}$$

5. Verify that each of the following differential equations is exact and then find the general solution.

(a) $2xy\, dx + \left(x^2 + 1\right) dy = 0$

- $$\begin{aligned} M &= 2xy \\ N &= x^2 + 1 \\ \frac{\partial M}{\partial y} &= 2x = \frac{\partial N}{\partial x} \quad \Rightarrow \quad Exact \end{aligned}$$

  Since the differential equation is exact it is equivalent to an algebraic relation of the form

  $$\phi(x, y) = C$$

  with

  $$\frac{\partial \phi}{\partial x} = M = 2xy \tag{3}$$

  $$\frac{\partial \phi}{\partial y} = N = x^2 + 1 \tag{4}$$

  The most general function ϕ satisfying (**??**) is obtained taking the anti-partial dervivative with respect to x; i.e., by integrating with respect to x, treating y as a constant, and allowing the possibility of an arbitrary function of y in the result:

  $$\phi(x, y) = \int \frac{\partial \phi}{\partial x} \partial x = \int (2xy) \partial x = yx^2 + H_1(y)$$

  Similarly, the most general function ϕ satisfying (**??**) is

  $$\phi(x, y) = \int \frac{\partial \phi}{\partial y} \partial y = \int (x^2 + 1) \partial y = x^2 y + y + H_2(x)$$

Comparing these two expressions for $\phi(x,y)$ and demanding that they agree with one another, we see that we must take

$$\begin{aligned} H_1(y) &= y \\ H_2(x) &= 0 \end{aligned}$$

Hence, $\phi(x,y) = x^2y + y$ and our differential equation is equivalent to the following family of algebraic relations

$$x^2y + y = C \quad , \quad \text{with } C \text{ an arbitrary constant .}$$

Solving this relation for y yields

$$y(x) = \frac{C}{x^2+1}.$$

(b) $3x^2y\,dx + \left(x^3+1\right)dy = 0$

- $$\begin{aligned} M &= 3x^2y \;\Rightarrow\; \frac{\partial M}{\partial y} = 3x^2 \\ N &= x^3+1 \;\Rightarrow\; \frac{\partial N}{\partial x} = 3x^2 \end{aligned}$$

  Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.

  $$\begin{aligned} \phi(x,y) &= \int \frac{\partial\phi}{\partial x}\partial x = \int M(x,y)\partial x = \int 3x^2y\partial x = x^3y + H_1(y) \\ \phi(x,y) &= \int \frac{\partial\phi}{\partial y}\partial y = \int N(x,y)\partial y = \int \left(x^3+1\right)\partial y = x^3y + y + H_2(x) \end{aligned}$$

  Comparing these two expressions for $\phi(x,y)$ we see that we must take $H_1(y) = y$ and $H_2(x) = 0$. So $\phi(x,y) = x^3y + y$ and the differential equation is equivalent to

  $$x^3y + y = C$$

  or

  $$y(x) = \frac{C}{1+x^3}$$

(c) $y(y+2x)dx + x(2y+x)dy = 0$

- $$\begin{aligned} M &= y^2 + 2yx \;\Rightarrow\; \frac{\partial M}{\partial y} = 2y + 2x \\ N &= 2yx + x^2 \;\Rightarrow\; \frac{\partial N}{\partial x} = 2y + 2x \end{aligned}$$

  Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.

  $$\begin{aligned} \phi(x,y) &= \int \frac{\partial\phi}{\partial x}\partial x = \int M(x,y)\partial x = \int (y^2+2yx)\partial x = y^2x + yx^2 + H_1(y) \\ \phi(x,y) &= \int \frac{\partial\phi}{\partial y}\partial y = \int N(x,y)\partial y = \int \left(2yx+x^2\right)\partial y = y^2x + x^2y + H_2(x) \end{aligned}$$

  Comparing these two expressions for $\phi(x,y)$ we see that we must take $H_1(y) = 0$ and $H_2(x) = 0$. So $\phi(x,y) = x^2y + y^2x$ and the differential equation is equivalent to

  $$x^2y + y^2x = C$$

  Solving this equation for y yields

  $$y = \frac{1}{2x}\left(-x^2 \pm \sqrt{(x^4+4xC)}\right).$$

(d) $y\cos(xy)\,dx + x\cos(xy)\,dy = 0$

- $$\begin{aligned} M &= y\cos(xy) \;\Rightarrow\; \frac{\partial M}{\partial y} = \cos(xy) - xy\sin(xy) \\ N &= x\cos(xy) \;\Rightarrow\; \frac{\partial N}{\partial x} = \cos(xy) - xy\sin(xy) \end{aligned}$$

  Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the equation is exact.

  $$\begin{aligned} \phi(x,y) &= \int \frac{\partial \phi}{\partial x}\partial x = \int M(x,y)\partial x = \int y\cos(xy)\partial x = \sin(xy) + H_1(y) \\ \phi(x,y) &= \int \frac{\partial \phi}{\partial y}\partial y = \int N(x,y)\partial y = \int x\cos(xy)\partial y = \sin(xy) + H_2(x) \end{aligned}$$

  Comparing these two expressions for $\phi(x,y)$ we see that we must take $H_1(y) = 0$, $H_2(x) = 0$, and $\phi(x,y) = \sin(xy)$. Thus the original differential equation is equivalent to

  $$\sin(xy) = C$$

  or

  $$y = \frac{C'}{x}$$

  (Here $C' = \sin^{-1}(C)$.)

6. Solve the following initial value problems.

(a) $(x - y\cos(x)) - \sin(x)y' = 0$, $y\left(\frac{\pi}{2}\right) = 1$

- This equation is exact since

  $$\frac{\partial}{\partial y}(x - y\cos(x)) = -\cos(x) = \frac{\partial}{\partial x}(\sin(x))$$

  Therefore, it must be equivalent to an algebraic equation of the form $\phi(x,y) = C$ with

  $$\begin{aligned} \phi(x,y) &= \int \frac{\partial \phi}{\partial x}\partial x = \int M(x,y)\partial x = \int (x - y\cos(x))\,\partial x = \frac{1}{2}x^2 - y\sin(x) + H_1(y) \\ \phi(x,y) &= \int \frac{\partial \phi}{\partial y}\partial y = \int N(x,y)\partial y = \int (-\sin(x))\,\partial y = -y\sin(x) + H_2(x) \end{aligned}$$

  Comparing these two expressions for $\phi(x,y)$ we see we must take $H_1(y) = 0$, $H_2(x) = \frac{1}{2}x^2$, and $\phi(x,y) = \frac{1}{2}x^2 - y\sin(x)$. Hence we must have

  $$\frac{1}{2}x^2 - y\sin(x) = C.$$

  Before solving for y we'll impose the initial condition: $x = \frac{\pi}{2} \Rightarrow y = 1$ to first determine C.

  $$C = \frac{1}{2}x^2 - y\sin(x) = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - (1)\sin(\frac{\pi}{2}) = \frac{1}{8}\pi^2 - 1.$$

  Now we solve for y :

  $$y = \frac{\frac{1}{2}x^2 - C}{\sin(x)} = \csc(x)\left(\frac{1}{2}x^2 + 1 - \frac{1}{8}\pi^2\right)$$

(b) $x^2 + y^2 + 2xyy' = 0$, $y(1) = 1$

- This equation is exact since

$$\frac{\partial}{\partial y}\left(x^2+y^2\right) = 2y = \frac{\partial}{\partial x}\left(2xy\right).$$

Therefore, the differential equation is equivalent to an algebraic relation of the form $\phi(x,y) = C$ with

$$\begin{aligned}\phi(x,y) &= \int \frac{\partial \phi}{\partial x}\partial x = \int M(x,y)\partial x = \int \left(x^2+y^2\right)\partial x = \frac{1}{3}x^3 - xy^2 + H_1(y) \\ \phi(x,y) &= \int \frac{\partial \phi}{\partial y}\partial y = \int N(x,y)\partial y = \int \left(2xy\right)\partial y = xy^2 + H_2(x)\end{aligned}$$

Comparing these two expressions for $\phi(x,y)$ we see we must take $H_1(y) = 0$, $H_2(x) = \frac{1}{3}x^3$, and so $\phi(x,y) = \frac{1}{3}x^3 - xy^2$. We thus have

$$\frac{1}{3}x^3 + xy^2 = C.$$

We now impose the initial condition $x = 1 \;\Rightarrow\; y = 1$ to fix C:

$$C = \frac{1}{3}x^3 + xy^2 = \frac{1}{3}(1)^3 + (1)(1)^2 = \frac{4}{3}.$$

Hence, the differential equation together with the initial condition implies that y must satisfy

$$\frac{1}{3}x^3 + xy^2 = \frac{4}{3}.$$

Solving this equation for y yields

$$y = \pm\sqrt{\frac{1}{3x}\left(4-x^3\right)}$$