Math 2233
Solutions to Homework Set 2

1. Solve the following differential equation using Separation of Variables.

$$\frac{dy}{dx} = xe^y$$

- We can explicitly separate the x-dependence from the y-dependence in this equation by multiplying both sides by $e^{-y}dx$:

$$e^{-y}dx\left(\frac{dy}{dx} = xe^y\right) \quad \Rightarrow \quad e^{-y}dy = xdx$$

Integrating both sides of the resulting equation yields

$$-e^{-y} = \int e^{-y}dy = \int xdx = \frac{1}{2}x^2 + C$$

or

$$e^{-y} = C' - \frac{1}{2}x^2$$

or

$$-y = \ln\left|C' - \frac{1}{2}x^2\right|$$

or

$$y(x) = -\ln\left|C' - \frac{1}{2}x^2\right| .$$

2. Solve the following differential equation using Separation of Variables.

$$\frac{dx}{dt} = txe^{t^2}$$

- Multiplying both sides of this equation by $\frac{1}{x}dt$ yields

$$\frac{dx}{x} = te^{t^2}dt$$

and so the equation is separable. Integrating both sides we get

$$\ln|x| = \int\frac{dx}{x} = \int te^{t^2}dt = \int\frac{1}{2}e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2} + C$$

where we have used the substitutions $u = t^2$, $du = 2tdt$, to carry out the integration over t. Solving the extreme sides of this equation for x yields

$$x(t) = \exp\left(\frac{1}{2}e^{t^2} + C\right) = \tilde{C}\exp\left(\frac{1}{2}e^{t^2}\right)$$

3. Solve the following differential equation using Separation of Variables.

$$x^2y' + e^y = 0$$

- Taking the e^y term to the right hand side and then multiplying by $x^{-2}e^{-y}dx$ yields

$$e^{-y}dy = -\frac{dx}{x^2}$$

Integrating both sides of this equation yields

$$-e^{-y} = \int e^{-y}dy = \int -\frac{dx}{x^2} = -\left(-\frac{1}{x}\right) + C$$

or

$$e^{-y} = -\frac{1}{x} - C$$

or

$$-y = \ln\left[-C - \frac{1}{x}\right]$$

or

$$y = -\ln\left[C' - \frac{1}{x}\right]$$

where we have just replace C by $C' = -C$ (so that we don't have to think too hard about how to take a logarithm of a negative number).

$$y(x) = \ln\left|C' - \frac{1}{x}\right| .$$

4. Solve the following differential equation using Separation of Variables.

$$yy' = e^x$$

- Multiplying both sides by dx yields

$$ydy = e^x dx.$$

Integrating both sides of this equation produces

$$\frac{1}{2}y^2 = \int ydy = \int e^x dx = e^x + C$$

Solving the extreme sides of this equation for y yields

$$y(x) = \pm\sqrt{2e^x + C'} .$$

5. Solve $y' + 3y = x + e^{-2x}$.

- This equation is already in standard form with

$$\begin{aligned} p(x) &= 3 \\ g(x) &= x + e^{-2x} \end{aligned}$$

We can calculate the solution to this first order linear ODE using the formula

$$y(x) = \frac{1}{\mu(x)} \int^x \mu(s) g(s) ds + \frac{C}{\mu(x)}$$

where

$$\mu(x) = \exp\left[\int^x p(x) ds\right] .$$

First, we calculate $\mu(x)$:

$$\mu(x) = \exp\left[\int^x (3) ds\right] = \exp[3x] = e^{3x}$$

Then we calculate $y(x)$:

$$\begin{aligned} y(x) &= \frac{1}{e^{3x}} \int^x e^{3s}\left(s + e^{-2s}\right) ds + \frac{C}{e^{3x}} \\ &= e^{-3x} \int^x \left(se^{3s} + e^s\right) ds + Ce^{-3x} \\ &= e^{-3x}\left(\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + e^x\right) + Ce^{-3x} \\ &= \frac{1}{3}x - \frac{1}{9} + e^{-2x} + Ce^{-3x} \end{aligned}$$

Thus

$$y(x) = \frac{1}{3}x - \frac{1}{9} + e^{-2x} + Ce^{-3x}$$

6. Solve

$$y' - y = 2e^x \quad .$$

- This is a first order linear ODE with

$$p(x) = -1 \ \ , \ \ g(x) = 2e^x \ .$$

So

$$\mu(x) = \exp\left[\int^x p(s)ds\right] = \exp\left[\int^x -ds\right] = \exp\left[-x\right] = e^{-x}$$

and

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^x \mu(s)g(s)ds + \frac{C}{\mu(x)} \\ &= \frac{1}{e^{-x}}\int^x e^{-s}\left(2e^s\right)ds + \frac{C}{e^{-x}} \\ &= e^x\int^x 2ds + Ce^x \\ &= 2xe^x + Ce^x \end{aligned}$$

Thus,

$$y(x) = 2xe^x + Ce^x \ .$$

7. Solve

$$xy' + 2y = \sin(x)$$

- This is another first order linear ODE. However, before we can apply our formular, we must correctly identify the functions $p(x)$ and $g(x)$. Dividing through by x we cast the differential equation into standard form

$$y' + \frac{2}{x}y = \frac{1}{x}\sin(x).$$

Hence,

$$p(x) = \frac{2}{x} \ \ , \ \ g(x) = \frac{1}{x}\sin(x) \ .$$

Now we can calculate $\mu(x)$:

$$\begin{aligned} \mu(x) &= \exp\left[\int^x p(s)ds\right] \\ &= \exp\left[\int^x \frac{2}{s}ds\right] \\ &= \exp\left[2\ln|x|\right] \\ &= \exp\left[\ln|x^2|\right] \\ &= x^2 \end{aligned}$$

And now that we have $\mu(x)$ we can calculate $y(x)$.

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int^x \mu(s)g(s)ds + \frac{C}{\mu(x)} \\ &= \frac{1}{x^2}\int^x s^2\left(\frac{1}{s}\sin(s)\right)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\int^x s\sin(s)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\left(-x\cos(x) + \sin(x)\right) + \frac{C}{x^2} \end{aligned}$$

Hence,

$$y(x) = -\frac{1}{x}\cos(x) + \frac{1}{x^2}\sin(x) + \frac{C}{x^2} \ .$$

8. Solve the following initial value problem

$$y' - y = 2xe^{2x} \qquad , \qquad y(1) = 0$$

- First, we'll find the general solution. This is first order, linear, ODE, and so we compute the integrating factor. Noting that this differential equation is already in the standard form $y' + p(x)\,y = g(x)$ with $p(x) = -1$ and $g(x) = 2xe^{2x}$, we first compute the integrating factor

$$\mu(x) = \exp\left(\int p(x)\right) = \exp\left(\int (-1)\,dx\right) = e^{-x}$$

Now we can compute the general solution as

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)} \int \mu(x)\,g(x)\,dx \quad + \quad \frac{C}{\mu(x)} = \frac{1}{e^{-x}} \int e^{-x}\left(2xe^{2x}\right)dx \quad + \quad \frac{C}{e^{-1}} \\ &= 2e^{x} \int xe^{x}dx \quad + \quad Ce^{x} \\ &= 2e^{x}\left(xe^{x} - e^{x}\right) \quad + \quad Ce^{x} \\ &= 2xe^{2x} - 2e^{2x} + Ce^{x} \end{aligned}$$

Next, we plug the general solution into the initial condition to determine the appropriate value of the constant C :

$$\begin{aligned} 0 &= y(1) = \left(2xe^{2x} - 2e^{2x} + Ce^{x}\right)\Big|_{x=1} = 2e^{2} - 2e^{2} + Ce \\ &= Ce \end{aligned}$$

We conclude C must equal 0 and so our solution is

$$y(x) = 2xe^{2x} - 2e^{2x}$$

9. Solve the following initial value problem.

$$y' + \frac{2}{x}y = \frac{\cos(x)}{x^2} \quad ; \quad y(\pi) = 0$$

- This is a first order linear ODE with $p(x) = \frac{2}{x}$ and $g(x) = \frac{\cos(x)}{x^2}$. Hence

$$\mu(x) = \exp\left[\int^{x} p(s)ds\right] = \exp\left[\int^{x} \frac{2}{s}ds\right] = \exp\left[2\ln|x|\right] = \exp\left[\ln|x^2|\right] = x^2$$

and so the general solution of the ODE is

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)} \int^{x} \mu(s)g(s)ds + \frac{C}{\mu(x)} \\ &= \frac{1}{x^2} \int^{x} s^2\left(\frac{\cos(s)}{s^2}\right)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2} \int^{x} \cos(s)ds + \frac{C}{x^2} \\ &= \frac{1}{x^2}\sin(x) + \frac{C}{x^2} \end{aligned}$$

We now impose the initial condition to fix C.

$$0 = y(\pi) = \frac{1}{\pi^2}\sin(\pi) + \frac{C}{\pi^2} = 0 + \frac{C}{\pi^2} = \frac{C}{\pi^2}$$

So we must take $C = 0$. The solution to the initial value problem is thus

$$y(x) = \frac{\sin(x)}{x^2}.$$