Math 2233
Solutions to Homework Set 1

1. Determine the order of the following differential equations, whether or not the equations are linear and whether the differential equations are ODEs (ordinary differential equations) or PDEs (partial differential equations).

(a) $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 2y = \sin(x)$

- This is a second order, linear, ODE.

(b) $(1+y^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = e^x$

- This is a second order, non-linear, ODE (the $y^2 \frac{d^2y}{dx^2}$ term makes it non-linear)

(c) $\dfrac{\partial^2\phi}{\partial x^2} + y^2 \dfrac{\partial \phi}{\partial x} = x^2$

- This is a second order, linear, PDE.

(d) $\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 1$

- This is a fourth order, linear, ODE.

(e) $\dfrac{dy}{dx} + xy^2 = 0$

- This is a first order, non-linear, ODE (the xy^2 term in non-linear in the *unknown function* y).

(f) $\dfrac{\partial^2\phi}{\partial x \partial y} + \dfrac{\partial \phi}{\partial x}\phi = x^2$

- This is a second order, non-linear, PDE (the term $\frac{\partial\phi}{\partial x}\phi$ is not simultaneously linear in ϕ and $\frac{\partial \phi}{\partial x}$)

(g) $\dfrac{d^2y}{dx^2} + \sin(x+y) = \sin(x)$

- This is a second order, non-linear, ODE (the function $\sin(x+y)$ is a non-linear function of y).

2.

(a) Plot the direction field for the differential equation

$$y' = x^2 + y.$$

- We begin by chosing a representative grid of points in the xy-plane. Here I'll consider the points (x, y) where $0 \leq x \leq 4$ and $0 \leq y \leq 4$ and I'll use step-sizes of $\Delta x = 0.5$ and $\Delta y = 0.5$. Thus, $x \in \{0.0, 0.5, 1.0.1.5, \ldots, 3.5, 4.0\}$ and $y \in \{0.0, 0.5, 1.0, \ldots, 3.5, 4.0\}$.

Next, I'll build a table with colums x, y, and $x^2 + y$. The last column represents the slope (as prescribed by the differential equation) of a solution that passes thru the corresponding point (x, y).

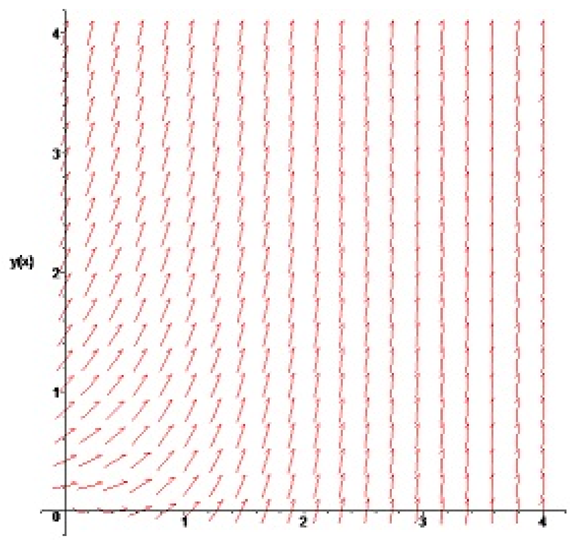
x	y	$x^2 + y$
0.0	0.0	0.0
0.0	0.5	0.5
0.0	1.0	1.0
0.0	1.5	1.5
0.0	2.0	2.0
0.0	2.5	2.5
0.0	3.0	3.0
0.0	3.5	3.5
0.0	4.0	5.0

x	y	$x^2 + y$
0.5	0.0	.25
0.5	0.5	.75
0.5	1.0	1.25
0.5	1.5	1.75
0.5	2.0	2.25
0.5	2.5	2.75
0.5	3.0	3.25
0.5	3.5	3.75
0.5	4.0	4.24

$\rightarrow \cdots \quad \rightarrow$

x	y	$x^2 + y$
4.0	0.0	16.0
4.0	0.5	16.5
4.0	1.0	17.0
4.0	1.5	17.5
4.0	2.0	18.0
4.0	2.5	18.5
4.0	3.0	19.0
4.0	3.5	19.5
4.0	4.0	20.0

The next step is to plot little arrows in the xy-plane from the data in this table. For each value of x and each value of y in your table, you plot a little arrow with slope equal to $x^3 + y$. Thus, for example, at the point with coordinates $(0, 5, 2.0)$ on your graph paper, you'd make a short arrow with slope 2.25. Below is a (computer generated) plot (done in essentially the same way, but with even more data points.

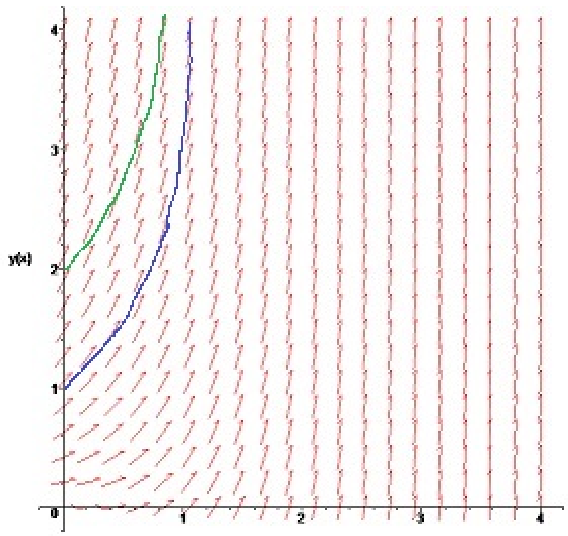


(b) Sketch the solution that satisfies $y(0) = 2$.

- To sketch a solution, you place our pencil on the direction field plot (produced above) at the point where the initial condition is defined and then try to sketch a curve that starts at that spot and follows the arrows of the plot. In our case, the initial condition says when $x = 0$, $y = 2$. so the initial point on the solution curve will be $(0, 2)$. Below I have indicated the corresponding solution on the plot as the green curve.

(c) Sketch the solution that satisfies $y(0) = 1$.

- This solution corresponds to the blue curve shown below. Note that it starts at the point $(0, 1)$



3. Consider the differential equation $y' = (y+2)*(y-2)$. What can you say about the behavior of solution $y(x)$ that passes through the point $x = 0$, $y = -1$ as $x \to \infty$? (Hint: the sign of the right hand side of the differential equations tells you whether or not a solution y(x) is increasing or decreasing.)

- By virtue of the differential equation

$$y' = (y+2)(y-2)$$

we see that the value of y determines whether a solution $y(x)$ is increasing, decreasing, or constant (that is to say, when $y'(x)$ is positive, negative, or zero).< 0 are decreasing.

(1) If $y > 2$, then $y' < 0$. This is because if $y > 2$ both factors on the right hand side of the differential equation are positive:

$$y > 2 \quad \Rightarrow \quad y+2 > 0 \quad \text{and} \quad y-2 > 0$$

So in this region, their product $(y+2)(y-2)$ will be positive and so solutions in the region have positive derivatives are therefore always decreasing.

(2) if $y = 2$, then $y' = 0$, and so solutions for which $y = 2$ are constant and so stick to the line $y = 2$.

(3) if $-2 < y < 2$, then $y' < 0$. This because in this region $y+2 > 0$ but $y-2 < 0$. Hence the derivative of a solution in this region must be negative. Thus, the solutions in the region $-2 < y < 2$ are decreasing.

(4) if $y = -2$, then $y' = 0$, and so solutions for which $y = -2$ are constant, and so stick to the line $y = -2$.

(5) if $y < -2$, then $y' > 0$, since both factors are positive. The solutions in this region will be increasing with x.

We thus have four basic classes of solutions.

- The solutions in the region $y > 2$ are always increasing. They will tend to ∞ as $x \to \infty$.
- The solutions that begin at $y = 2$ will stay on the line $y = 2$.
- The solutions in the region $-2 < y < 2$ are always decreasing. However, they cannot decrease past the line $y = -2$ as they approach that line their derivatives must tend to zero. This means the solutions curves flatten out as they approach $y = -2$. In this region, the solutions approach the line $y = -2$ *asymptotically* as $x \to \infty$.
- The solutions that begiat at $y = -2$ will stay on the line $y = -2$.
- The solutions in the region $y < -2$ are always increasing. However, as they approach the line $y = -2$, they must level out since the derivatives when $y = -2$ are zero. These solutions must tend to the value -2 as $x \to \infty$.

4. Using the Euler Method, find an approximate value for $y(1)$ for the following initial value problem (take $h = \Delta x = 0.2$):

$$\frac{dy}{dx} = x + y \quad , \quad y(0) = 1$$

- We'll do this problem by hand. In accordance with the initial condition $y(0) = 1$ we set $x_0 = 0$ and $y_0 = 1$. To get the next pair of points on the solution curve we use the fact that the slope of the best straight line fit to the solution curve at $(x_0, y_0) = (0, 1)$ must be

$$m_0 = \left.\frac{dy}{dx}\right|_{(x_0,y_0)} = x_0 + y_0 = 0 + 1 = 1.$$

  Setting

$$x_1 = x_0 + \Delta x = 0 + .2 = 0.2$$

  we get an approximate value for $y_1 = y(x_1)$ using the formula $\Delta y = m\Delta x$; (for the case at hand, this formula implies $y_1 = y_0 + m_0\Delta x$)

$$\begin{aligned} y_1 &= y_0 + m_0\Delta x \\ &= 1 + (1)(0.2) \\ &= 1.2 \end{aligned}$$

  Thus the next pair of points on the solution curve should be $(x_1, y_1) = (0.2, 1.2)$. Now we calculate the slope of the best strainght line fit the to solution that passes through the point (x_1, y_1):

$$m_1 = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = x_1 + y_1 = 0.2 + 1.2 = 1.4$$

  Taking $x_2 = x_1 + \Delta x = 0.4$, we calculate y_2

$$\begin{aligned} y_2 &= y_1 + m_1\Delta x \\ &= 1.2 + (1.4)(0.2) \\ &= 1.48 \end{aligned}$$

  We continue in this manner:

$$\begin{aligned} m_2 &= x_2 + y_2 = 0.4 + 1.48 = 1.88 \\ x_3 &= x_2 + \Delta x = 0.4 + 0.2 = 0.6 \\ y_3 &= y_2 + m_2\Delta x = 1.48 + (1.88)(0.2) = 1.856 \\ m_3 &= x_3 + y_3 = 0.6 + 1.856 = 2.456 \\ x_4 &= x_3 + \Delta x = 0.6 + 0.2 = 0.8 \\ y_4 &= y_3 + m_3\Delta x = 1.856 + (2.456)(0.2) = 2.3472 \\ m_4 &= x_4 + y_4 = 0.8 + 2.3472 = 3.1472 \\ x_5 &= x_4 + \Delta x = 0.8 + 0.2 \\ y_5 &= y_4 + m_4\Delta x = 2.3472 + (3.1472)(0.2) = 2.9766 \end{aligned}$$

  Thus $y(1) = y(x_5) = y_5 = 2.9766$.

5. Using the Euler Method, find an approximate value for $y(1.5)$ for the following initial value problem (take $h = \Delta x = 0.1$):

$$\frac{dy}{dx} = xe^y \quad , \quad y(1) = 0$$

- This problem is solved liked its predecessor. The initial values of x and y are given by

$$\begin{aligned} x_0 &= 1.0 \\ y_0 &= 0.0 \end{aligned}$$

We determine successive values of x and y via

$$\begin{aligned} x_{i+1} &= x_i + 0.1 \\ y_{i+1} &= y_i + \left(x_i e^{y_i}\right)(0.1) \end{aligned}$$

On then finds

$$\begin{aligned} x_1 &= 1.1 \\ y_1 &= 0.1 \\ &\downarrow \\ x_2 &= 1.2 \\ y_2 &= 0.221569 \\ &\downarrow \\ x_3 &= 1.4 \\ y_3 &= 0.371333 \\ &\downarrow \\ x_4 &= 1.4 \\ y_4 &= 0.550789 \\ &\downarrow \\ x_5 &= 1.5 \\ y_5 &= 0.804832 \end{aligned}$$

Thus, the approximate value for $y(1.5)$ will be 0.804832.