**Math 2233**
SOLUTION TO SAMPLE SECOND EXAM
Summer 2017

1. Given that $y_1(x) = x^{-1}$ and $y_2(x) = x^2$ are solutions to $x^2 y'' - 2y = 0$
(a) (5 pts) Show that the functions $y_1(x)$ and $y_2(x)$ are linearly independent.

- $$W[y_1, y_2] = \left(x^{-1}\right)\left(x^2\right)' - \left(x^{-1}\right)'\left(x^2\right) = \left(x^{-1}\right)(2x) - \left(-x^{-2}\right)\left(x^2\right) = 2 + 1 = 3 \neq 0$$

  Since the Wronskian $W[y_1, y_2] \neq 0$, the functions $y_1$ and $y_2$ are linearly independent.

$\square$

(b) (5 pts) Write down the general solution.

- $$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x^{-1} + c_2 x^2$$

$\square$

(c) (5 pts) Find the solution satisfying the initial conditions $y(1) = 2$, $y'(1) = 1$.

- For our general solution $y(x) = c_1 x^{-1} + c_2 x^2 \quad \Longrightarrow \quad y'(x) = -c_1 x^{-2} + 2c_2 x$ and

  $$\left.\begin{array}{l} 2 = y(1) = c_1 + c_2 \\ 1 = y'(1) = -c_1 + 2c_2 \end{array}\right\} \Longrightarrow c_1 = 1 \quad \text{and} \quad c_2 = 1$$

  So

  $$y(x) = x^{-1} + x^2 \quad .$$

$\square$

2. (10 pts) Given that $y_1(x) = x^{-2}$ is one solution of $x^2 y'' + 5xy' + 4y = 0$, use Reduction of Order to determine the general solution.

- The differential equation, when cast in standard form is $y'' + (5/x)y' + (4/x^2)y = 0$, and so $p(x) = 5/x$. Plugging this and $y_1(x) = x^{-2}$ into the Reduction of Order formula, yields

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{(y_1)^2} \exp\left(-\int^x p\,dx'\right) dx \\ &= x^{-1} \int \frac{1}{(x^{-2})^2} \exp\left(-\int^x \frac{5}{x'} dx'\right) dx = x^{-1} \int x^4 \exp(-5\ln(x))\, dx = x^{-1} \int \left(x^4\right)\left(x^{-5}\right) dx \\ &= x^{-1} \int x^{-1} dx = x^{-1} \ln|x| \end{aligned}$$

  The general solution is thus

  $$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x^{-2} + c_2 x^{-2} \ln|x| \quad .$$

$\square$

3. (10 pts) Explain in words and formulas how you would construct the general solution of $y'' + p(x)y' + q(x)y = g(x)$, given that $y_1(x)$ is a solution of $y'' + p(x)y' + q(x)y = 0$. (That is, describe the general procedure, writing down the relevant formulas. It is **not** necessary to carry out any calculations.)

- First, we could use Reduction of Order to calculate a second independent solution of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$ :

  $$y_1 \int \frac{1}{(y_1)^2} \exp\left(-\int^x p\,dx'\right) dx \quad .$$

- Second, with $y_1(x)$, $y_2(x)$ and $g(x)$ in hand, we could apply the Variation of Parameters formula to get a particular solution of the inhomogeneous equation $y'' + p(x)y' + q(x)y = g(x)$.

  $$y_p(x) = -y_1(x) \int \frac{y_2(x)\, g(x)}{W[y_1, y_2](x)} dx + y_2(x) \int \frac{y_1(x)\, g(x)}{W[y_1, y_2](x)} dx \quad .$$

- Lastly, with a particular solution $y_p(x)$ of $y'' + p(x)\, y' + q(x)\, y = g(x)$ and a pair $y_1(x)$, $y_2(x)$ of independent solutions of $y'' + p(x)\, y' + q(x)\, y = 0$ in hand, we can write down the general solution of $y'' + p(x)\, y' + q(x)\, y = g(x)$ as

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) \quad .$$

$\square$

4. Determine the general solution of the following differential equations.

(a) (5 pts) $y'' - 3y' - y = 0$

- This ODE is second order, linear, with constant coefficients. The characteristic equation

$$0 = \lambda^2 - 3\lambda - 1 \quad \Longrightarrow \quad \lambda = \frac{3 \pm \sqrt{9+4}}{2} = \frac{3}{2} \pm \frac{\sqrt{13}}{2}$$

has two real roots, $\lambda_+ = \frac{3}{2} + \frac{\sqrt{13}}{2}$ and $\lambda_- = \frac{3}{2} - \frac{\sqrt{13}}{2}$. The general solution is thus

$$y(x) = c_1 e^{\frac{1}{2}(3+\sqrt{13})x} + c_2 e^{\frac{1}{2}(3-\sqrt{13})x}$$

$\square$

(b) (5 pts) $4y'' - 4y' + y = 0$

- This ODE is second order, linear, with constant coefficients. The characteristic equation

$$0 = 4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 \quad \Longrightarrow \quad \lambda = \frac{1}{2}$$

has a single real root : $\lambda = -\frac{1}{2}$. The general solution is thus

$$y(x) = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$$

$\square$

(c) (5 pts) $y'' + 4y' + 13y = 0$

- This ODE is second order, linear, with constant coefficients. The characteristic equation

$$0 = \lambda^2 + 4\lambda + 13 \quad \Longrightarrow \quad \lambda = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i$$

has a pair of complex roots: $\lambda = -2 \pm 3i$. The general solution is thus

$$y(x) = c_1 e^{-2x}\cos(3x) + c_2 e^{-2x}\sin(3x) \quad .$$

$\square$

(d) (5 pts) $x^2y'' + 3xy' - 8y = 0$

- This is an Euler type differential equation. Its indicial equation

$$0 = m(m-1) + 3m - 8 = m^2 + 2m - 8 = (m+4)(m-2) \quad \Longrightarrow \quad m = -4, 2$$

has two real roots. The general solution is thus

$$y(x) = c_1 x^{-4} + c_2 x^2$$

$\square$

(e) (5 pts) $x^2y'' - 3xy' + 4y = 0$

- This is an Euler type differential equation. Its indicial equation

$$0 = m(m-1) - 3m + 4 = m^2 - 4m + 4 = (m-2)^2 \quad \Longrightarrow \quad m = 2$$

has a single real root. The general solution is thus

$$y(x) = c_1 x^2 + c_2 x^2 \ln|x|$$

$\square$

(f) (5 pts) $x^2y'' - 2xy' + 3y = 0$

- This is an Euler type differential equation. Its indicial equation

$$0 = m(m-1) - 2m + 3 = m^2 - 3m + 3 \quad \Longrightarrow \quad m = \frac{3 \pm \sqrt{9-12}}{2} = \frac{3}{2} \pm \frac{\sqrt{3}}{2}i$$

has a pair of complex roots. The general solution is thus

$$y(x) = c_1 x^{3/2}\cos\left(\frac{\sqrt{3}}{2}\ln|x|\right) + c_2 x^{3/2}\sin\left(\frac{\sqrt{3}}{2}\ln|x|\right)$$

$\square$

5. Given that $y_1(x) = e^x$ and $y_2(x) = e^{-3x}$ are solutions of $y'' + 2y' - 3y = 0$.
(a) (10 pts) Use the Method of Variation of Parameters to find a particular solution of $y''+2y'-3y = e^{2x}$ .

- We have $g(x) = e^{2x}$ and $W[y_1, y_2] = (e^x)\left(e^{-3x}\right)' - (e^x)'\left(e^{-3x}\right) = (e^x)\left(-3e^{-3x}\right) - (e^x)\left(e^{-3x}\right) = -4e^{-2x}$, and so

$$y_p(x) = -y_1 \int \frac{y_2(x)\,g(x)}{W[y_1,y_2](x)}dx + y_2 \int \frac{y_1(x)\,g(x)}{W[y_1,y_2](x)}dx = -e^x \int \frac{\left(e^{-3x}\right)\left(e^{2x}\right)}{-4e^{-2x}}dx + e^{-3x} \int \frac{\left(e^{x}\right)\left(e^{2x}\right)}{-4e^{-2x}}dx$$
$$= \frac{1}{4}e^x \int e^x dx - \frac{1}{4}e^{-3x} \int e^{5x} dx = \frac{1}{4}e^x\left(e^x\right) - \frac{1}{4}e^{-3x}\left(\frac{1}{5}e^{5x}\right) = \frac{1}{4}\left(1 - \frac{1}{5}\right)e^{2x} = \frac{1}{5}e^{2x}$$

$\square$

(b) (5 pts) Find the solution of the differential equation in part (a) satisfying $y(0) = 0$, $y'(0) = 2$.

- The general solution is

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) = \frac{1}{5}e^{2x} + c_1 e^x + c_2 e^{-3x} \quad \Longrightarrow \quad y'(x) = \frac{2}{5}e^{2x} + c_1 e^x - 3c_2 e^{-3x}$$

Plugging into the initial conditions yields

$$\left.\begin{array}{l} 0 = y(0) = \frac{1}{5} + c_1 + c_2 \\ 2 = y'(0) = \frac{2}{5} + c_1 - 3c_2 \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} c_1 = \frac{1}{4} \\ c_2 = -\frac{9}{20} \end{array}\right. \quad \Longrightarrow \quad y(x) = \frac{1}{5}e^{2x} + \frac{1}{4}e^x - \frac{9}{20}e^{-3x}$$

$\square$