Math 2233
SOLUTIONS TO SECOND EXAM
July 19, 2017

1. Given that $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions to $y'' - 2y' + y = 0$

(a) (5 pts) Show that the functions $y_1(x)$ and $y_2(x)$ are linearly independent.

- $$W[y_1, y_2] \equiv y_1 y_2' - y_1' y_2 = (e^x)(e^x + xe^x) - (e^x)(xe^x) = e^{2x} \neq 0 \quad \Rightarrow \quad y_1, y_2 \text{ are independent}$$

  Alternatively, $y_1(x) \neq \lambda y_2(x) \quad \Rightarrow \quad W[y_1, y_2] \neq 0 \quad \Rightarrow \quad y_1, y_2$ are independent

(b) (5 pts) Write down the general solution.

- $$y(x) = c_1 e^x + c_2 x e^x$$

(c) (5 pts) Find the solution satisfying the initial conditions $y(1) = 1$, $y'(1) = 2$.

- Applying the initial conditions to the general solution yields

  $$1 = y(1) = c_1 e^1 + c_2 (1) e^1 = ec_1 + ec_2$$
  $$2 = y'(1) = \left.(c_1 e^x + c_2 e^x + c_2 x e^x)\right|_{x=1} = ec_1 + 2ec_2$$

  Subtracting the second equation from the first yields $-1 = -ec_2 \quad \Rightarrow \quad c_2 = e^{-1}$. Inserting this value for c_2 in the first equation yields $1 = ec_1 + 1 \quad \Rightarrow \quad c_1 = 0$. Thus, $c_1 = 0$ and $c_2 = 1/e$; hence

  $$y(x) = \frac{1}{e} x e^x$$

2. (10 pts) Given that $y_1(x) = x^3$ is one solution of $x^2 y'' - 5xy' + 9y = 0$, use Reduction of Order to determine the general solution.

- We use Reduction of Order to calculate a second independent solution. Before plugging into the Reduction of Order formula, we note that the differential equation in *standard form* is $y'' - \frac{5}{x} y' + \frac{9}{x^2} y = 0$; and so $p(x) = -5/x$.

  $$y_2 = y_1 \int \frac{1}{(y_1)^2} \exp\left(-\int p dx\right) dx = x^3 \int \frac{1}{x^6} \exp\left(+\int \frac{5}{x} dx\right) = x^3 \int \frac{1}{x^6} \exp(5 \ln|x|)$$
  $$= x^3 \int \frac{x^5}{x^6} dx = x^3 \ln|x|$$

  $y_1(x) = x^3$ and $y_2(x) = x^3 \ln|x|$ are two independent solutions of the homogeneous linear differential equation; hence its general solution is

  $$y(x) = c_1 x^3 + c_2 x^3 \ln|x|$$

3. (15 pts) Explain in words and formulas how you would construct the general solution of $y'' + p(x) y' + q(x) y = g(x)$, given that $y_1(x)$ is a solution of $y'' + p(x) y' + q(x) y = 0$. (That is, describe the general procedure, writing down the relevant formulas. It is **not** necessary to carry out any calculations.)

- Step 1: Use the Reduction of Order formula $y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left(-\int p(x) dx\right) dx$ to calculate a second independent solution, $y_2(x)$, of the homogeneous differential equation $y'' + p(x) y' + q(x) y = 0$.
- Step 2: Plug $y_1(x)$ and $y_2(x)$ into the Variation of Parameters formula $Y_p(x) = -y_1(x) \int \frac{y_2(x) g(x)}{W[y_1, y_2](x)} dx + y_2(x) \int \frac{y_1(x) g(x)}{W[y_1, y_2](x)} dx$ to calculate a particular solution of the inhomogeneous differential equation $y'' + p(x) y' + q(x) y = g(x)$.
- Write down the general solution of the inhomogeneous differential equation as

  $$Y(x) = Y_p(x) + c_1 y_1(x) + c_2 y_2(x) \qquad (c_1, c_2 \text{ being arbitrary constants})$$

4. Determine the general solution of the following differential equations.

(a) (5 pts) $y'' - 5y' + 6y = 0$

- A constant coefficients differential equation. Substituting $y(x) = e^{\lambda x}$ into the differential equation yields the following *characteristic equation* for λ

$$0 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) \quad \Rightarrow \quad \lambda = 2, 3$$

Thus, e^{2x} and e^{3x} will be two independent solutions and the general solution will be

$$y(x) = c_1 e^{2x} + c_2 e^{3x}$$

(b) (5 pts) $y'' - 6y' + 9y = 0$

- A constant coefficients differential equation. Substituting $y(x) = e^{\lambda x}$ into the differential equation yields the following *characteristic equation* for λ

$$0 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \quad \Rightarrow \quad \lambda = 3 \text{ (only one root)}$$

In this situation we have only one exponential solution $y_1(x) = e^{3x}$. A second solution solution can be found by Reduction of Order, but in the contant coefficient case with a single root, that second solution always turns out to be $y_2(x) = xy_1(x) = xe^{3x}$. The general solution is thus

$$y(x) = c_1 e^{3x} + c_2 x e^{3x}$$

(c) (5 pts) $y'' - 6y' + 13y = 0$

- A constant coefficients differential equation. Substituting $y(x) = e^{\lambda x}$ into the differential equation yields the following *characteristic equation* for λ

$$0 = \lambda^2 - 6\lambda + 13 \quad \Rightarrow \quad \lambda = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$$

In this situation, the general solution can be written by plugging the real (α) and imaginary (β) parts of the complex roots $\lambda = \alpha \pm i\beta$ into the following template for the geneal solution

$$y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x) = c_1 e^{3x} \cos(2x) + c_2 e^{3x} \sin(2x)$$

(d) (5 pts) $x^2 y'' + 3xy' - 8y = 0$

- An Euler type ODE. Substituting $y(x) = x^m$ into the differential equation yields the following *indicial equation* for m:

$$0 = m(m-1) + 3m - 8 = m^2 + 2m - 8 = (m+4)(m-2) \quad \Rightarrow \quad m = -4, 2$$

We thus have two real roots $m = -4, 2$, two independent solutions $y_1(x) = x^{-4}$ and $y_2 = x^2$ and the following general solution

$$y(x) = c_1 x^{-4} + c_2 x^2$$

(e) (5 pts) $x^2 y'' - 3xy' + 4y = 0$

- An Euler type ODE. Substituting $y(x) = x^m$ into the differential equation yields the following *indicial equation* for m:

$$0 = m(m-1) - 3m + 4 = m^2 - 4m + 4 = (m-2)^2 \quad \Rightarrow \quad m = 2 \qquad \text{(one real root)}$$

In this situation, we have one solution $y_1(x) = x^2$. A second independent solution can be found via Reduction of Order; but in the current situation (an Euler type equation with a single root), the Reduction of Order calculation just yields a second solution of the form $y_2(x) = y_1(x) \ln|x| = x^2 \ln|x|$. Thus, the general solution will be

$$y(x) = c_1 x^2 + c_2 x^2 \ln|x|$$

(f) (5 pts) $x^2 y'' + 3xy' + 10y = 0$

- An Euler type ODE. Substituting $y(x) = x^m$ into the differential equation yields the following *indicial equation* for m:

$$0 = m(m-1) + 3m + 10 = m^2 + 2m + 10 \quad \Rightarrow \quad m = \frac{-2 \pm \sqrt{4 - 40}}{2} = \frac{-2 \pm \sqrt{-36}}{2}$$

or $m = 1 \pm 3i$. Since we have a pair of complex roots, we can write down the general solution by substituting $\alpha = 1 = \operatorname{Re}(m)$ and $\beta = 3 = \operatorname{Im}(m)$ into the following template:

$$y(x) = c_1 x^{\alpha} \cos(\beta \ln|x|) + c_2 x^{\alpha} \sin(\beta \ln|x|) = c_1 x^{-1} \cos(3 \ln|x|) + c_2 x^{-1} \sin(2 \ln|x|)$$

5. Given that $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$ are solutions of $y'' - 5y' + 6y = 0$,
(a) (10 pts) Use the Method of Variation of Parameters to find a particular solution of $y'' - 5y' + 6y = e^x$

- The differential equation is in standard form with $g(x) = e^x$. The Wronskian of y_1 and y_2 is

$$W[y_1, y_2] = \left(e^{2x}\right)\left(3e^{3x}\right) - \left(2e^{2x}\right)\left(e^{3x}\right) = 3e^{5x} - 23^{5x} = e^{5x}$$

We can now plug into the Variation of Parameters formula:

$$\begin{aligned} Y_p(x) &= -y_1(x)\int \frac{y_2(x)\,g(x)}{W[y_1,y_2](x)}dx + y_2(x)\int \frac{y_1(x)\,g(x)}{W[y_1,y_2](x)}dx \\ &= -e^{2x}\int \frac{e^{3x}e^x}{e^{5x}}dx + e^{3x}\int \frac{e^{2x}e^x}{e^{5x}}dx = -e^{2x}\int e^{-x}dx + e^{3x}\int e^{-2x}dx \\ &= -e^{2x}\left(-e^{-x}\right) + e^{3x}\left(-\frac{1}{2}e^{-2x}\right) \\ &= e^x - \frac{1}{2}e^x = \frac{1}{2}e^x \end{aligned}$$

(b) (5 pts) Find the solution of $y'' - 5y' + 6y = e^x$ satisfying $y(0) = 0, y'(0) = 1$.

- With the particular solution $Y_p(x) = \frac{1}{2}e^x$ in hand, along with the two independent solutions $y_1(x) = e^{2x}$, $y_2(x) = e^{3x}$ of the corresponding homogeneous problem, we can immediately write down the general solution of the inhomogeneous differential equation

$$Y(x) = Y_p(x) + c_1y_1(x) + c_2y_2(x) = \frac{1}{2}e^x + c_2e^{2x} + c_2e^{3x}$$

We next impose the initial conditions on the general solution:

$$\begin{aligned} 0 = y(0) &= \frac{1}{2}e^0 + c_1e^0 + c_2e^0 = \frac{1}{2} + c_1 + c_2 \quad \Rightarrow \quad c_1 + c_2 = -\frac{1}{2} \\ 1 = y'(0) &= \left[\frac{1}{2}e^x + 2c_1e^{2x} + 3c_2e^{3x}\right]\Big|_{x=0} = \frac{1}{2} + 2c_1 + 3c_2 \quad \Rightarrow \quad 2c_1 + 3c_2 = \frac{1}{2} \end{aligned}$$

Solving this pair of equations for c_1, c_2 yields $c_1 = -2$ and $c_2 = \frac{3}{2}$. Thus, the solution is

$$y(x) = \frac{1}{2}e^x - 2e^{2x} + \frac{3}{2}e^{3x}$$

6. (15 pts) Given that $y(x) = c_1x^2 + c_2x^3$ is the geneal solution of $x^2y'' - 4xy' + 6y = 0$; find the general solution of $x^2y'' - 4xy + 6y = x^3$.

- We'll first employ Variation of Parameters to find a particular solution $y_p(x)$ of the inhomogeneous differential equation, which once put in standard form is

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x \quad \Rightarrow \quad g(x) = x$$

We can read off two independent solutions $y_1(x) = x^2$ and $y_2(x) = x^3$ from the form of the general solution of the corresponding homogeneous ODE. Their Wronskian is

$$W[y_1, y_2] = \left(x^2\right)\left(3x^2\right) - (2x)\left(x^3\right) = x^4$$

We now plug into the Variation of Parameters formula

$$\begin{aligned} y_p(x) &= -y_1(x)\int \frac{y_2(x)\,g(x)}{W[y_1,y_2](x)}dx + y_2(x)\int \frac{y_1(x)\,g(x)}{W[y_1,y_2](x)}dx \\ &= -x^2\int \frac{\left(x^3\right)(x)}{x^4}dx + x^3\int \frac{\left(x^2\right)(x)}{x^4}dx = -x^2\int dx + x^3\int \frac{1}{x}dx \\ &= -x^2(x) + x^2\ln|x| \\ &= -x^3 + x^3\ln|x| \end{aligned}$$

With $y_p(x)$, $y_1(x)$ and $y_2(x)$ in hand, we can now write down the general solution of the given inhomogeneous ODE

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) = -x^3 + x^3\ln|x| + c_1x^2 + c_2x^3$$

By redefining the arbitary constant c_1 ($c_1 \to c_1 + 1$), this general solution could be written more simply as

$$y(x) = x^3 \ln|x| + c_1 x^2 + c_2 x^3$$