Math 2233
FIRST EXAM SOLUTIONS
June 29, 2017

1. (15 pts) Classify the following differential equations: determine their order, if they are linear or non-linear, and if they are ordinary differential equations or partial differential equations.

(a) $y'' + \cos(y) = x$
• 2^{nd} order, nonlinear, ODE

(b) $\dfrac{\partial \Phi}{\partial y} + \dfrac{\partial^2 \phi}{\partial x^2} = y^2$
• 2^{nd} order, linear, PDE

(c) $\dfrac{d^3x}{dt^3} + x^2\dfrac{dx}{dt} + x = 0$
• 3^{rd} order, non-linear, ODE

(d) $a\left(x\right)y' + b\left(x\right)y + c\left(x\right) = 0$
• 1^{st} order, linear, ODE

(e) $\dfrac{dx}{dt} = x^2$
• 1^{st} order, nonlinear, ODE

2. (10 pts) Consider the following first order ODE: $y' = x + y$ and suppose $y(x)$ is the solution satisfying $y(1) = 1$. Use the numerical (Euler) method with $n = 3$ and $\Delta x = 0.1$ to estimate $y(1.3)$.

• We will begin constructina a table of approximate values for points $(x_i, y_i \approx y\left(x_i\right))$ on the solution using the Euler formula

$$\begin{aligned} x_{i+1} &= x_i + \Delta x \\ y_{i+1} &= y_i + F\left(x_i, y_i\right)\Delta x \end{aligned}$$

with $F(x,y) = x + y$ and $x_0 = 1$, $y_0 = 1$.

$$\begin{array}{ll} x_1 = x_0 + \Delta x = 1.1 & y_1 = y_0 + m\left(x_0, y_0\right)\Delta x = y_0 + \left(x_0 + y_0\right)\Delta x = 1 + (1+1)(0.1) = 1.2 \\ x_2 = x_1 + \Delta x = 1.2 & y_2 = y_1 + m\left(x_1, y_1\right)\Delta x = y_1 + \left(x_1 + y_1\right)\Delta x = 1.2 + (1.1 + 1.2)(0.1) = 1.43 \\ x_3 = x_2 + \Delta x & y_3 = y_2 + m\left(x_2, y_2\right)\Delta x = y_2 + \left(x_2 + y_2\right)\Delta x = 1.43 + (1.2 + 1.43)(0.1) = 1.693 \end{array}$$

So $y(1.3) \approx 1.693$.

□

3. (15 pts) Find an explicit solution of the following (separable) differential equation.

$$2x - e^{2y}y' = 0$$

- We have $M(x) = 2x$ and $N(y) = -e^{2y}$, as an implicit solution we'll have

$$\int 2x dx - \int e^{2y} dy = C \qquad \Rightarrow \qquad x^2 - \frac{1}{2}e^{2y} = C$$

Solving for y we obtain

$$y = \frac{1}{2}\ln\left|2x^2 - 2C\right|$$

□

4. (15 pts) Solve the following initial value problem

$$y' - \frac{3}{x}y = x \qquad , \qquad y(1) = 2$$

- This is a first order linear equation with $p(x) = -3/x$ and $g(x) = x$. So the general solution is

$$\begin{aligned}
\mu(x) &= \exp\left(\int p(x)dx\right) = \exp\left(\int -\frac{3}{x}dx\right) = \exp\left(-3\ln|x|\right) = x^{-3} \\
y(x) &= \frac{1}{\mu}\int \mu g dx + \frac{C}{\mu} = \frac{1}{x^{-3}}\int x^{-3}(x)\, dx + \frac{C}{x^{-3}} = x^3\int x^{-2}dx + Cx^{-3} \\
&= x^3\left(\frac{1}{-1}x^{-1}\right) + Cx^3 = -x^2 + Cx^3
\end{aligned}$$

Plugging the general solution into the initial condition yields

$$\begin{aligned}
2 &= y(1) = \left[-x^2 + Cx^3\right]\big|_{x=1} = -1 + C \qquad \Rightarrow \qquad C = 3 \\
\Rightarrow & \qquad y = -x^2 + 3x^3
\end{aligned}$$

□

5. (15 pts) Show that the following equation is exact.

$$\frac{y}{x} + 2x + \ln|x|\frac{dy}{dx} = 0$$

and then fine the explicit solution of this differential equation.

- For this problem, we have $M(x, y) = y/x + 2x$ and $N(x, y) = \ln|y|$. We have

$$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}$$

so the equation is exact. Let's now find an explicit solution to the following initial value problem

$$\frac{y}{x} + 2x + \ln|x|\frac{dy}{dx} = 0$$

$$\begin{aligned}
\Phi(x, y) &= \int M\partial x + C_1(y) = \int\left(\frac{y}{x} + 2x\right)\partial x + C_1(y) = y\ln|x| + x^2 + C_1(y) \\
&= \int N\partial y + C_2(x) = \int \ln|x|\,\partial y + C_2(x) = \ln|x|\, y + C_2(x)
\end{aligned}$$

The consistency for these two expression for Φ requires $C_1(y) = 0$ and $C_2(x) = x^2$. Thus, $\Phi = y\ln|x| + x^2$. Our implicit solution is thus

$$y\ln|x| + x^2 = C \qquad \Rightarrow \qquad y = \frac{C - x^2}{\ln|x|}$$

□

6. (15 pts) Find an integrating factor for the following equation

$$1 + 2xe^{-2y} + 2x\frac{dy}{dx} = 0$$

(Hint Look for an integrating factor depending only on y.):

- Since we are told to expect an integrating factor depending only on y, we look at

$$F_2 = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{1+2xe^{-2y}}\left(2 - \left[0 + 2x(-2e^{-2y})\right]\right) = \frac{2\left(1+2xe^{-2y}\right)}{1+2xe^{-2y}} = 2$$

This does not depend on x, so

$$\mu(y) = \exp\left(\int F_2(y)dy\right) = \exp\left(\int 2dy\right) = \exp(2y) = e^{2y}$$

will be our integrating factor.

$\square$

7. (15 pts) Use a change of variables to find the general solution of

$$\frac{dy}{dx} = \frac{x^2+2y^2}{2xy}$$

(Hint: this equation is homogeneous of degree 0, so try $z = y/x$.)

- If we set $z = y/x$, we'll have

$$y = zx \quad \Rightarrow \quad y' = z'x + z$$

If we replace y by zx on the right hand side of the differential equation and y' by $z'x + z$ on the left hand side we get

$$z'x + z = \frac{x^2 + 2(zx)^2}{2x(zx)} = \frac{x^2\left(1+2z^2\right)}{2x^2 z} = \frac{1+2z^2}{2z} = \frac{1}{2z} + z$$

Canceling the isolated z terms from both extreme sides we obtain

$$z'x = \frac{1}{2z} \quad \Rightarrow \quad 2zz' = \frac{1}{x}$$

This last equation is separable, and hence easy to solve

$$\int 2zdz \quad = \quad \int \frac{1}{x}dx + C \qquad \Rightarrow \quad z^2 = \ln|x| + C \qquad \Rightarrow \quad z = \pm\sqrt{\ln|x| + C}$$

$$\Rightarrow \qquad y = \pm x\sqrt{\ln|x| + C}$$