1. (3 pts) Consider the plot below of the direction field for the differential equation \( y' = -(y - 1)(y + 1) \).

![Direction Field Image]

Sketch the solution curve satisfying \( y(0) = 0 \).

2. (12 pts) Classify the following differential equations: determine their order, if they are linear or non-linear, and if they are ordinary differential equations or partial differential equations.

(a) \( y'' + \cos(y) = x \)
- \( 2^{nd} \) order, non-linear, ODE

(b) \( \frac{\partial \Phi}{\partial y} + \frac{\partial^2 \Phi}{\partial x^2} = y^2 \)
- \( 2^{nd} \) order, linear, PDE

(c) \( \frac{d^3x}{dt^3} + x^2 \frac{dx}{dt} + x = 0 \)
- \( 3^{rd} \) order, non-linear, ODE

(d) \( a(x)y' + b(x)y + c(x) = 0 \)
- \( 1^{st} \) order, linear, ODE

3. (10 pts) Consider the following first order ODE: \( y' = x + y \) and suppose \( y(x) \) is the solution satisfying \( y(1) = 1 \). Use the numerical (Euler) method with \( n = 3 \) and \( \Delta x = 0.1 \) to estimate \( y(1.3) \).

\[
\begin{align*}
x_0 & = 1 & y_0 & = 1 \\
x_1 & = x_0 + \Delta x = 1.1 & y_1 & = y_0 + m(x_0,y_0) \Delta x = y_0 + (x_0 + y_0) \Delta x = 1 + (1 + 1)(0.1) = 1.2 \\
x_2 & = x_1 + \Delta x = 1.2 & y_2 & = y_1 + m(x_1,y_1) \Delta x = y_1 + (x_1 + y_1) \Delta x = 1.2 + (1.1 + 1.2)(0.1) = 1.43 \\
x_3 & = x_2 + \Delta x & y_3 & = y_2 + m(x_2,y_2) \Delta x = y_2 + (x_2 + y_2) \Delta x = 1.43 + (1.2 + 1.43)(0.1) = 1.693 \\
\end{align*}
\]

So \( y(1.3) \approx 1.693 \).
4. (10 pts) Consider the following nonlinear first order ODE: \( y' = xy^2 \). Write down the first four terms of the Taylor expansion of the solution satisfying \( y(1) = 1 \) about \( x = 1 \) (i.e. the terms up to order \((x - 1)^3\)).

- We have
  \[
  \begin{align*}
  y(1) &= 1 \\
  y'(1) &= xy^2|_{x=1} = (1)(1)^2 = 1 \\
  y''(1) &= \frac{d}{dx}xy^2|_{x=1} = [y^2 + 2xyy']|_{x=1} = 1^2 + (2)(1)(1) = 3 \\
  y'''(1) &= \frac{d}{dx}(y^2 + 2xyy')|_{x=1} = [4yy' + 2x(y')^2 + 2xyy'']|_{x=1} = 4(1)(1) + 2(1)(1)^2 + 2(1)(1)(3) = 12
  
  \end{align*}
  \]

  So
  \[
  y(x) = y(1) + y'(1)(x - 1) + \frac{1}{2}y''(1)(x - 1)^2 + \frac{1}{6}y'''(1)(x - 1)^3 + \mathcal{O}(|x - 1|^4)
  \]
  \[
  \approx 1 + (x - 1) + \frac{3}{2} (x - 1)^2 + 2(x - 1)^3
  \]

5. (10 pts) Find an explicit solution of the following (separable) differential equation.
\[
2x - e^{2y}y' = 0
\]

- We have \( M(x) = 2x \) and \( N(y) = -e^{2y} \), as an implicit solution we'll have
  \[
  \int 2xdx - \int e^{2y}dy = C \quad \Rightarrow \quad x^2 - \frac{1}{2}e^{2y} = C
  \]
  Solving for \( y \) we obtain
  \[
  y = \frac{1}{2} \ln |2x^2 - 2C|
  \]
6. (15 pts) Solve the following initial value problem

\[ y' - \frac{3}{x}y = x, \quad y(1) = 2 \]

- This is a first order linear equation with \( p(x) = -\frac{3}{x} \) and \( g(x) = x \). So the general solution is

\[
\begin{align*}
\mu(x) &= \exp \left( \int p(x) \, dx \right) = \exp \left( \int -\frac{3}{x} \, dx \right) = \exp (-3 \ln |x|) = x^{-3} \\
y(x) &= \frac{1}{\mu} \int \mu g \, dx + C = \frac{1}{x^{-3}} \int x^{-3} (x) \, dx + C = x^3 \int x^{-2} \, dx + C x^{-3} \\
&= x^3 \left( \frac{1}{-1} x^{-1} \right) + C x^3 = -x^2 + C x^3
\end{align*}
\]

Plugging the general solution into the initial condition yields

\[
2 = y(1) = \left[ -x^2 + C x^3 \right]_{x=1} = -1 + C \quad \Rightarrow \quad C = 3
\]

7. (a) (5 pts) Show that the following equation is exact.

\[ \frac{y}{x} + 2x + \ln |x| \frac{dy}{dx} = 0 \]

- For this problem, we have \( M(x, y) = \frac{y}{x} + 2x \) and \( N(x, y) = \ln |y| \). We have

\[
\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}
\]

so the equation is exact.

(b) (10 pts) Find an explicit solution to the following initial value problem

\[ \frac{y}{x} + 2x + \ln |x| \frac{dy}{dx} = 0, \quad y(1) = 1 \]

- \begin{align*}
\Phi(x, y) &= \int M \, dx + C_1(y) = \int \left( \frac{y}{x} + 2x \right) \, dx + C_1(y) = y \ln |x| + x^2 + C_1(y) \\
&= \int N \, dy + C_2(x) = \int \ln |x| \, dy + C_2(x) = \ln |y| + C_2(x)
\end{align*}

The consistency for these two expressions for \( \Phi \) requires \( C_1(y) = 0 \) and \( C_2(x) = x^2 \). Thus, \( \Phi = y \ln |x| + x^2 \). Our implicit solution is thus

\[ y \ln |x| + x^2 = C \]

Applying the initial condition we can fix \( C \):

\[ C = (1) \ln |1| + 1^2 = 0 + 1 \quad \Rightarrow \quad C = 1 \]

We now solve the implicit solution for \( y \):

\[ y \ln |x| + x^2 = 1 \quad \Rightarrow \quad y = \frac{1 - x^2}{\ln |x|} \]

\[ \square \]
8. (10 pts) Find an integrating factor for the following equation

\[ 1 + 2xe^{-2y} + 2x \frac{dy}{dx} = 0 \]

(Hint: Look for an integrating factor depending only on \( y \).):

- Since we are told to expect an integrating factor depending only on \( y \), we look at

\[ F_2 = \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{1 + 2xe^{-2y}} \left( 2 - \left[ 0 + 2x(-2e^{-2y}) \right] \right) = \frac{2 \left( 1 + 2xe^{-2y} \right)}{1 + 2xe^{-2y}} = 2 \]

This does not depend on \( x \), so

\[ \mu(y) = \exp \left( \int F_2(y) \, dy \right) = \exp \left( \int 2 \, dy \right) = \exp (2y) = e^{2y} \]

will be our integrating factor.

9. (15 pts) Use a change of variables to find the general solution of

\[ \frac{dy}{dx} = \frac{x^2 + 2y^2}{2xy} \]

(Hint: this equation is homogeneous of degree 0, so try \( z = y/x \).)

- If we set \( z = y/x \), we'll have

\[ y = zx \quad \Rightarrow \quad y' = z'x + z \]

If we replace \( y \) by \( zx \) on the right hand side of the differential equation and \( y' \) by \( z'x + z \) on the left hand side we get

\[ z'x + z = \frac{x^2 + 2(zx)^2}{2x(zx)} = \frac{x^2 (1 + 2z^2)}{2x^2 z} = \frac{1 + 2z^2}{2z} = \frac{1}{2z} + z \]

Canceling the isolated \( z \) terms from both extreme sides we obtain

\[ z'x = \frac{1}{2z} \quad \Rightarrow \quad 2z z' = \frac{1}{x} \]

This last equation is separable, and hence easy to solve

\[ \int 2z \, dz = \int \frac{1}{x} \, dx + C \quad \Rightarrow \quad z^2 = \ln |x| + C \quad \Rightarrow \quad z = \pm \sqrt{\ln |x| + C} \]

\[ y = \pm x \sqrt{\ln |x| + C} \]