1. Consider the plot below of the direction field for the differential equation $y' = -(y - 1)(y + 1)$.

(a) (5 pts) Sketch the solution curve satisfying $y(0) = 0$.

(b) (5 pts) Suppose $y(x)$ is a solution satisfying $y(100) = -1.1$. What can you say about the asymptotic behavior of $y(x)$ as $x \to \infty$? Justify your answer using only the differential equation (Hint: think about when a solution must be increasing, decreasing or constant.)

- We have $y' = -(y - 1)(y + 1) < 0$ whenever $y < -1$. This means that if a solution $y(x)$ begins in the region where $y < -1$ the solution will start decreasing and continue to decrease. Hence

$$\lim_{x \to \infty} y(x) = -\infty$$

2. (10 pts) Classify the following differential equations: determine their order, if they are linear or non-linear, and if they are ordinary differential equations or partial differential equations.

(a) $y'' + \cos(y) = x$ : $2^{nd}$ order, nonlinear, ODE

(b) $\frac{\partial \Phi}{\partial y} + \frac{\partial^2 \Phi}{\partial x^2} = y^2$ : $2^{nd}$ order, linear, PDE

(c) $\frac{d^3 x}{dt^3} + x^2 \frac{dx}{dt} + x = 0$ : $3^{rd}$ order, nonlinear, ODE

(d) $a(x) y' + b(x) y + c(x) = 0$ : $1^{st}$ order, linear, ODE

(e) $\frac{\partial^2 \Psi}{\partial x^2} + \left( \frac{\partial \Psi}{\partial x} \right) \Psi = xy^2$ : $2^{nd}$ order, nonlinear, PDE
3. (10 pts) Consider the following first order ODE: \( y' = x^2 + y \) and suppose \( y(x) \) is the solution satisfying \( y(1) = 1 \). Use the numerical (Euler) method with \( n = 3 \) and \( \Delta x = 0.1 \) to estimate \( y(1.2) \).

\[
\begin{align*}
x_0 &= 1 \\
y_0 &= 1 \\
x_1 &= x_0 + \Delta x = 1.1 \\
y_1 &= y_0 + F(x_0, y_0) \Delta x = 1 + (1^2 + 1) \cdot (0.1) = 1.2 \\
x_2 &= x_1 + \Delta (x) = 1.2 \\
y_2 &= y_1 + F(x_1, y_1) \Delta x = 1.2 + \left((1.1)^2 + 1.2\right)(0.1) = 1.2 + (1.121 + .12) = 1.441
\end{align*}
\]

\[\implies y(1.2) \approx 1.441\]

4. (10 pts) Consider the following nonlinear first order ODE: \( y' = x^2y^2 \). Write down the first four terms of the Taylor expansion of the solution satisfying \( y(1) = 1 \) about \( x = 1 \) (i.e. the terms up to order \( (x-1)^3 \)).

\[
\begin{align*}
y(1) &= 1 \\
y'(1) &= x^2y^2\big|_{x=1} = 1^2(y(1))^2 = 1 \\
y''(1) &= \frac{d}{dx} y\big|_{x=1} = \frac{d}{dx} x^2 y^2 \big|_{x=1} = (2xy^2 + 2x^2yy')\big|_{x=1} = 2(1)(1)^2 + 2(1^2)(1)(1) = 4 \\
y'''(1) &= \frac{d}{dx} y''\big|_{x=1} = \frac{d}{dx} (2x^2y^2+2x^2yy')\big|_{x=1} = (2y^2+4xyy'+4xyy'+2x^2y'y'+2x^2yy'')\big|_{x=1} \\
&= 2(1)^2 + 4(1)(1)(1) + 4(1)(1)(1) + 2(1)^2(1)(1) + 2(1)^2(1)(4) = 20
\end{align*}
\]

So

\[
y(x) = y(1) + y'(1)(x-1) + \frac{1}{2}y''(1)(x-1)^2 + \frac{1}{6}y'''(1)(x-1)^3 + \cdots
\]

\[= 1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{20}{6}(x-1)^3 + \cdots\]

5. (10 pts) Find an explicit solution of the following (separable) differential equation.

\[
\cos(2x) - e^y y' = 0
\]

\[\text{We have } M(x) = \cos(2x) \text{ and } N(y) = e^y.\] So

\[
C = \int M(x) \, dx + \int N(y) \, dy = \int \cos(2x) \, dx - \int e^y \, dy = \frac{1}{2} \sin(2x) - e^y
\]

Solving for \( y \) we obtain

\[
y = \ln \left[ \frac{1}{2} \sin(2x) - C \right]
\]
6. (15 pts) Solve the following initial value problem
\[ xy' - 3y = x^2 \quad , \quad y(1) = 2 \]
- This is a 1st order linear differential equation with \( p(x) = -3/x \) and \( g(x) = x \). Thus,
  \[
  \mu (x) = \exp \left( \int p(x) \, dx \right) = \exp \left( - \int \frac{3}{x} \, dx \right) = \exp (-3 \ln |x|) = x^{-3}
  \]
  \[
  y (x) = \frac{1}{\mu (x)} \int \mu (x) g(x) \, dx + \frac{C}{\mu (x)} = \frac{1}{x^{-3}} \int (x^{-3}) (x) \, dx + \frac{C}{x^{-3}}
  \]
  \[
  = x^3 \int x^{-2} \, dx + C x^3 = x^3 (-x^{-1}) + C x^3
  \]
  \[
  = -x^2 + C x^3
  \]
- Plugging into the initial condition we find
  \[
  2 = y (1) = -(1)^2 + C (1)^3 \quad \Rightarrow \quad C = 3
  \]
- so the solution of the initial value problem is
  \[
  y (x) = -x^2 + 3x^3
  \]

7. (a) (5 pts) Show that the following equation is exact.
\[
\frac{y}{x} + 2x + \ln |x| \frac{dy}{dx} = 0
\]
- We have \( M (x, y) = y/x + 2x \) and \( N (x, y) = \ln |x| \) and
  \[
  \frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}
  \]
- so the equation is exact.

(b) (10 pts) Find the explicit solution to the differential equation in part (a).
- \[
  \Phi_1 = \int M (x, y) \, dx + C_1 (y) = \int \left( \frac{y}{x} + 2x \right) \, dx + C_1 (y) = y \ln |x| + x^2 + C_1 (y)
  \]
  \[
  \Phi_2 = \int N (x, y) \, dy + C_2 (x) = \int \ln |x| \, dy + C_2 (x) = \ln |x| y + C_2 (x)
  \]
- To get these two expressions for \( \Phi \) to agree we must take \( C_1 (y) = 0 \) and \( C_2 (z) = x^2 \). Thus, our implicit solution will be
  \[
  C = \Phi (x, y) = y \ln |x| + x^2
  \]
- Solving this for \( y \) yields the following explicit solution
  \[
  y (x) = \frac{C - x^2}{\ln |x|}
  \]
8. (10 pts) Find an integrating factor for
\[ y + e^{y-x} + 1 + xe^{y-x} \frac{dy}{dx} = 0 \]

- We have \( M(x, y) = y + e^{y-x} \) and \( N(x, y) = 1 + xe^{y-x} \). We look for an integrating factor depending only on \( x \). Since
\[
F_1 = \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{1 + xe^{y-x}} (1 + e^{y-x} - e^{y-x} + xe^{y-x}) = \frac{1 + xe^{y-x}}{1 + xe^{y-x}} = 1
\]
does not depend on \( y \),
\[
\mu(x) = \exp \left[ \int F_1(x) \, dx \right] = \exp \left[ \int dx \right] = \exp(x) = e^x
\]
will be an integrating factor.

9. (10 pts) Use a change of variables to solve
\[ \frac{dy}{dx} = \frac{x^2 + 2y^2}{2xy} \]

(Hint: this equation is homogeneous of degree 0.)

- We change variables to \( z = \frac{y}{x} \) \( \Rightarrow \) \( y = zx \) \( \Rightarrow \) \( y' = xz' + z \)
\[
xz' + z = y' = \frac{x^2 + 2y^2}{2xy} = \frac{1 + 2 \left( \frac{y}{x} \right)^2}{2 \left( \frac{y}{x} \right)} = \frac{1 + 2z^2}{2z} = \frac{1}{2z} + z
\]
or
\[
xz' = \frac{1}{2z} \Rightarrow 2zz' = \frac{1}{x}
\]
This last equation is separable.
\[
\int 2zdz = \int \frac{1}{x} \, dx + C \quad \Rightarrow \quad z^2 = \ln|x| + C \quad \Rightarrow \quad \left( \frac{y}{x} \right)^2 = \ln|x| + C
\]
or
\[
y = \pm\sqrt{x^2 \ln|x| + Cx^2}
\]