LECTURE 31

Laplace Transforms and Piecewise Continuous Functions

We have seen how one can use Laplace transform methods to solve 2^{nd} order linear Diff E's with constant coefficients, and have even pointed out some advantages of the Laplace transform technique over our original method of solving inhomogenous boundary value problems (where we first solved the characteristic equation to find two independent solutions y_1 and y_2 of the corresponding homogenous equation, then used Variation of Parameters to get a particular solution y_p of the inhomogenenous equation, and finally plugged $y = y_p + c_1y_1 + c_2y_2$ into the initial conditions to obtain the correct choice of c_1 and c_2).

Another big advantage is that the Laplace transform technique allows us to solve Diff E's of the form

$$ay'' + by' + cy = g(x)$$

where $g(x)$ is only a piecewise continuous function.

Theorem 31.1. *Suppose that*

(i) *f is a piecewise continuous function on the interval $[0, A]$ for any positive $A \subset \mathbb{R}$.*
(ii) *Suppose there exist positive constants M and K such that $|f(x)| \leq Ke^{at}$ when $0 > M$.*

Then the Laplace transform

$$\mathcal{L}[f](s) = \int_0^\infty f(x)\, e^{-sx} dx$$

exists for all $s > a$.

Example 31.2. Step functions.

Let c be a positive number and let $u_c(t)$ be the piecewise continuous function defined by

$$u_c(x) = \begin{cases} 0 & \text{if} \quad x < c \\ 1 & \text{if} \quad x \geq c \end{cases}$$

According to the theorem above $u_c(t)$ should have a Laplace transform for all $s \in [0, \infty)$; for evidently, if we can take $K = 2$ and $M = 1$, then

$$u_c(x) \leq 1 < Ke^{ax} = 2e^{0x} = 2 \quad \text{for all } x > 1$$

But this will also be apparent for the computation below

$$
\begin{aligned}
\mathcal{L}[u_c] &= \int_0^\infty u_c(x)\, e^{-sx} dx \\
&= \int_0^c u_c(x)\, e^{-sx} dx + \int_c^\infty u_c(x)\, e^{-sx} dx \\
&= 0 + \int_c^\infty e^{-sx} dx \\
&= \lim_{N\to\infty} \left.\left(\frac{1}{-s} e^{-sx}\right)\right|_c^N \\
&= 0 + \frac{e^{-cs}}{s} \\
&= \frac{e^{-cs}}{s}
\end{aligned}
$$

THEOREM 31.3. *If $F(s) = \mathcal{L}[f(x)]$ exists for $s > a \geq 0$, and if c is a positive constant, then*

$$\mathcal{L}[u_c(x) f(x-c)] = e^{-cs} F(s) \qquad , \qquad s > a$$

Conversely, if $f(x) = \mathcal{L}^{-1}[F(s)]$ then

$$u_c(x)\, f(x-c) = \mathcal{L}^{-1}\left[e^{-cs} F(s)\right]$$

THEOREM 31.4. *If $F(s) = \mathcal{L}[f(x)]$ exists for $s > a \geq 0$, and if c is a positive constant, then*

$$\mathcal{L}[e^{cx} f(x)] = F(s-c)$$

Conversely, if $f(x) = \mathcal{L}^{-1}[F(s)]$ then

$$e^{cx} f(x) = \mathcal{L}^{-1}[F(s-c)]$$

DEFINITION 31.5. *Suppose $f(x)$ is a function with the property that, for some fixed constant T,*

$$f(x+T) = f(x) \text{ for all } x$$

Then we say that f is a ***periodic function*** *with* ***period*** *T.*

THEOREM 31.6. *If $f(x)$ is periodic with period T, then*

$$\mathcal{L}[f(x)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-sx} f(x)\, dx$$

Proof.

$$
\begin{aligned}
\mathcal{L}[f(x)] &= \int_0^\infty e^{-sx} f(x)\,dx \\
&= \int_0^T e^{-sx} f(x)\,dx + \int_T^\infty e^{-sx} f(x)\,dx \\
&= \int_0^T e^{-sx} f(x)\,dx + \int_T^{2T} e^{-sx} f(x)\,dx + \int_{2T}^{3T} e^{-sx} f(x)\,dx + \cdots \\
&= \int_0^T e^{-sx} f(x)\,dx + \int_0^T e^{-s(x+T)} f(x+T)\,dx + \int_0^T e^{-s(x+2T)} f(x+2T)\,dx + \cdots \\
\text{(after changing variables } x &\rightarrow x+T \quad , \quad x \rightarrow x+2T \quad , \quad \textit{etc.}) \\
&= \int_0^T e^{-sx} f(x)\,dx + e^{-sT}\int_0^T e^{-sx} f(x+T)\,dx + e^{-2sT}\int_0^T e^{-sx} f(x+2T)\,dx + \cdots \\
&= \int_0^T e^{-sx} f(x)\,dx + e^{-sT}\int_0^T e^{-sx} f(x)\,dx + e^{-2sT}\int_0^T e^{-sx} f(x)\,dx + \cdots \\
\text{(using the fact that } f(x) &= f(x+T) = f(x+2T) = \cdots) \\
&= \left(1 + e^{-sT} + \left(e^{-sT}\right)^2 + \left(e^{-sT}\right)^3 + \cdots\right)\int_0^T e^{-sx} f(x)\,dx \\
&= \frac{1}{1+e^{-sT}}\int_0^T e^{-sx} f(x)\,dx \\
\text{(using the identity } \frac{1}{1-X} &= 1 + X + X^2 + X^3 + \cdots)
\end{aligned}
$$

1. Solving Differential Equations with Discontinous Driving Functions

EXAMPLE 31.7. Find the solution of

$$
\begin{aligned}
y'' + y &= f(t) = \begin{cases} 1 & , \quad 0 \leq t \leq \frac{\pi}{2} \\ 0 & , \quad \frac{\pi}{2} < t < \infty \end{cases} \\
y(0) &= 0 \\
y'(0) &= 0
\end{aligned}
$$

(This might correspond to a simple harmonic oscillator that was initially jolted by a constant force for $\frac{\pi}{2}$ seconds, and left alone.)

Notice that the driving function $f(t)$ is just

$$f(t) = u_0(t) - u_{\pi/2}(t)$$

Hence the Laplace transform of $f(t)$ will be

$$\mathcal{L}[f] = \mathcal{L}[u_0] - \mathcal{L}\left[u_{\pi/2}\right] = \frac{e^{-0s}}{s} - \frac{e^{-\frac{\pi s}{2}}}{s} = \frac{1}{s}\left(1 - e^{-\frac{\pi s}{2}}\right)$$

So the Laplace transform of the differential equation will be

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + \mathcal{L}[y] = \frac{1}{s}\left(1 - e^{-\frac{\pi s}{2}}\right)$$

or

$$\left(s^2+1\right)\mathcal{L}[y] = \frac{1}{s}\left(1 - e^{-\frac{\pi s}{2}}\right)$$

or

$$\begin{aligned}\mathcal{L}[y] &= \frac{1}{s(s^2+1)}\left(1-e^{-\frac{\pi s}{2}}\right)\\ &= \frac{1}{s(s^2+1)} - \frac{e^{-\frac{\pi s}{2}}}{s(s^2+1)}\end{aligned}$$

Now

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \quad\Rightarrow\quad 1 = A\left(s^2+1\right) + Bs^2 + Cs \quad\Rightarrow\quad \begin{cases} A=1\\ B=-1\\ C=0\end{cases}$$

So

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} = \mathcal{L}[1] + \mathcal{L}[\cos(t)] = \mathcal{L}[1+\cos(t)]$$

or

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1+\cos(t)$$

On the other hand, according to Theorem 31.3, if $\mathcal{L}[f] = F(s)$, then

$$\mathcal{L}^{-1}\left[e^{-cs}F(s)\right] = u_c(x)\,f(x)$$

so

$$\mathcal{L}^{-1}\left[\frac{e^{-\frac{\pi s}{2}}}{s(s^2+1)}\right] = u_{\pi/2}(t)\,\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = u_{\pi/2}(t)\,(1+\cos(t))$$

Thus,

$$\begin{aligned} y &= \mathcal{L}^{-1}[\mathcal{L}[y]]\\ &= \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)} - \frac{e^{-\frac{\pi s}{2}}}{s(s^2+1)}\right]\\ &= \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] - \mathcal{L}^{-1}\left[\frac{e^{-\frac{\pi s}{2}}}{s(s^2+1)}\right]\\ &= 1+\cos(t) - u_{\pi/2}(t)\,(1+\cos(t))\\ &= \left(1-u_{\pi/2}(t)\right)(1+\cos(t))\end{aligned}$$

2. Impulse Functions - The Dirac Delta Function

We have seen the Laplace transform technique is very good for solving differential equations

$$ay'' + by' + cy = g(x)$$

when the "driving function" $g(s)$ is only piecewise continuous. Physically such a differential equation might arise if an oscillatory system were given an initial push, or a recurrent push. But what happens when an oscillatory system is struck by a hammer?

To discuss such situations we first need a measure of how much energy is transferred to the system after the application of a constant force. As a crude measure of the amount of energy imparted to a system driven by a force $g(t)$ we introduce the **total impulse** I_g defined by

$$I_g \equiv \int_{-\infty}^{+\infty} g(t)\,dt$$

The way to think about this quantity is as follows: if $g(t)$ corresponds to the force applied to the system at time t, then I_g is the aggregate force applied to system over all time. Note how the magnitude of I_g depends not only on the magnitude of $g(t)$, but also on how long the force was applied (i.e. how long $g(t)$ is non-zero).

Now consider the following four driving functions

$$\begin{aligned}
g_1(t) &= \begin{cases} \frac{1}{4} & , \quad -2 \le t \le 2 \\ 0 & , \quad |t| > 2 \end{cases} & \Rightarrow \quad I_{g_1} = 1 \\
g_2(t) &= \begin{cases} \frac{1}{2} & , \quad -1 \le t \le 1 \\ 0 & , \quad |t| > 1 \end{cases} & \Rightarrow \quad I_{g_2} = 1 \\
g_3(t) &= \begin{cases} 1 & , \quad -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & , \quad |t| > \frac{1}{2} \end{cases} & \Rightarrow \quad I_{g_3} = 1 \\
g_4(t) &= \begin{cases} 2 & , \quad -\frac{1}{4} \le t \le \frac{1}{4} \\ 0 & , \quad |t| > \frac{1}{4} \end{cases} & \Rightarrow \quad I_{g_4} = 1
\end{aligned}$$

Note how the total impulse are all the same. More generally, if we set

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau} & , \quad -\tau \le t \le \tau \\ 0 & , \quad |t| > \tau \end{cases}$$

Then

$$I_{d_\tau} = \int_{-\infty}^{+\infty} d_\tau(t)\, dt = \int_{-\tau}^{\tau} \frac{1}{2\tau} dt = \left. \frac{t}{2\tau} \right|_{-\tau}^{\tau} = \frac{1}{2\tau}(\tau - (-\tau)) = \frac{2\tau}{2\tau} = 1$$

So all the driving functions $d_\tau(t)$, $\tau \in \mathbb{R}^+$ deliver the same total impulse, $I_{d_\tau} = 1$. We use this sort of driving functions to model situations like a hammer strike. For in such situations we want the duration of the force to be nearly instantaneous, yet we want a finite amount of energy to be transferred to the system. In fact, the situation we would really like to handle is the case where all the energy is transferred at a single instant $t = 0$. For this we would need something like

$$g(t) = \lim_{\tau \to 0} d_\tau(t)$$

However, there is no limit to $d_\tau(t)$ as $\tau \to 0$, since $\lim_{\tau \to 0} d_\tau(0) = \lim_{\tau \to 0} \frac{1}{2\tau} = \pm\infty$. Evidently, the family of unit impulse functions functions $d_\tau(t)$, $\tau > 0$, fails to converge to a function as $\tau \to 0$.

The surprising fact (at least at first) is that even though

$$\lim_{\tau \to 0} d_\tau$$

does not exist, its integral from $-\infty$ to $+\infty$ does: because

$$\lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t)\, dt = \lim_{\tau \to 0} 1 = 1$$

In fact, if $f(x)$ is any continuous function on the real line

$$\lim_{\tau \to 0} \int_{-\infty}^{+\infty} d_\tau(t)\, f(t)\, dt = f(0)$$

In this sense, we define the **Dirac delta function** $\delta(t)$

$$\delta(t) = \lim_{\tau \to 0} d_\tau(t)$$

with the understanding that it is not really a proper function, but nevertheless it has the property that when integrated from $-\infty$ to $+\infty$ against any function $f(t)$ the result is $f(0)$:

$$\int_{-\infty}^{+\infty} \delta(t)\, f(t)\, dt = f(0)$$

Now here's another surprising fact, $\delta(t)$ is not a function, but nevertheless it can still be differentiated (at least formally), so long as we keep it inside an integral. To evaluate

$$\int_{-\infty}^{+\infty} \delta'(t)\, f(t) dt$$

we simply apply integration by parts; using the integration by parts formula $\int udv = uv - \int vdu$ and the identifications

$$\begin{aligned} u &= f(t) \quad , \qquad dv = \delta'(t)\,dt \\ du &= f'(t)\,dt \quad , \qquad v = \delta(t) \end{aligned}$$

we have

$$\int_{-\infty}^{+\infty} \delta'(t)\,f(t)dt = f(t)\,\delta(t)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta(t)\,f'(t)\,dt$$

evaluation of $f(t)\,\delta(t)$ at the endpoints of integration yields 0 because $\delta(t) = 0$ for all $t \neq 0$, on the other hand, from the definition of $\delta(t)$

$$\int_{-\infty}^{+\infty} \delta(t)\,f'(t)\,dt = f'(0)$$

Thus,

$$\int_{-\infty}^{+\infty} \delta'(t)\,f(t)dt = -f'(0)\,.$$

and so $\delta'(t)$ is the (generalized) function that when integrated against a function $f(t)$, yields $-f'(0)$

EXAMPLE 31.8. Solve the following initial value problem:

$$\begin{aligned} y'' + 2y' + 2y &= \delta(t-1) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

(You can imagine this initial value problem as corresponding to a damped harmonic oscillator, initially at rest, and then struck by a hammer at time $t = 1$.

Taking the Laplace transform of the differential equation we get

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2(s\mathcal{L}[y] - y(0)) + 2\mathcal{L}[y] = \mathcal{L}[\delta(t-1)] = e^{-s}$$

or

$$\left(s^2 + 2s + 2\right)\mathcal{L}[y] = e^{-s} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{e^{-s}}{s^2 + 2s + 2}$$

Now

$$\frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1} = \mathcal{L}\left(e^{-t}\sin(t)\right)$$

Applying Theorem 31.3, viz,

$$\text{If } F(s) = \mathcal{L}[f] \quad \text{, then } \mathcal{L}^{-1}\left(e^{-cs}F(s)\right) = u_c(t)\,f(t-c)$$

we have

$$\begin{aligned} \mathcal{L}^{-1}\left(e^{-s}\frac{1}{(s+1)^2 + 1}\right) &= u_1(t)\,\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2 + 1}\right)(t-1) = u_1(t)\,e^{-(t-1)}\sin(t-1) \\ &= \begin{cases} 0 & \text{if } t < 1 \\ e^{-t+1}\sin(t-1) & \text{if t} \geq 1 \end{cases} \end{aligned}$$