LECTURE 28

Series Solutions about Regular Singular Points

Recall x_1 is a **singular point** for a differential equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

if

$$\lim_{x \to x_1} p(x) \text{ does not exist}$$

or

$$\lim_{x \to x_1} q(x) \text{ does not exist}$$

(Such singular points typically occur when $p(x)$ or $q(x)$ has a denominator that goes to 0 as $x \to x_1$). Recall also that a singular point x_1 is said to be a **regular singular point** if both

$$\lim_{x \to x_1} (x - x_1)\, p(x)$$

$$\lim_{x \to x_1} (x - x_1)^2\, q(x)$$

exist. (The above condition is equivalent to the one of the preceding lecture where defined x_1 to be a regular singular point if the degree of the singularity of $p(x)$ at x_1 was ≤ 1 and the degree of the singularity of $q(x)$ at x_1 was ≤ 2.)

In this lecture, I will show how the power series technique can be generalized to give (generalized power) series solutions that are defined (and computable) right up to (but not necessarily including) a regular singular point. (For singular points that are not regular, we have no such technique.)

1. Example: Bessel's equation

The following partial differential equation, Laplace's equation,

$$\boldsymbol{\nabla}^2 \phi := \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

is fundamental to many applications; ranging from electrostatics, to steady-state diffusion problems.

If one converts to polar coordinates r and θ where

$$r = \sqrt{x^2 + y^2} \quad , \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Laplace's equation becomes

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{1}$$

A standard technique for solving such a PDE is called *Separation of Variables*. In this technique, one looks for solutions of (1) of the form $\phi(r, \theta, z) = R(r)\,\Theta(\theta)\,Z(z)$; that is, a solution of (1) that factors into a product of a function R that only depends on the coordinate r alone, times a function Θ that only depends on θ alone, times a function Z that only depends on z. After plugging this *ansatz* for ϕ into the PDE (1), doing some algebra, and making a rather simple observation, the original PDE is seen to be equivalent to a system of 3 weakly coupled[1] ordinary differential equations:

$$\begin{aligned}
\frac{d^2 Z}{dz^2} &= \left(n^2 - m^2\right) Z \\
\frac{d^2 \Theta}{d\theta^2} &= -m^2 \Theta \\
r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + \left(n^2 r^2 - m^2\right) R &= 0
\end{aligned}$$

The first two equations are just second order linear ODEs with constant coefficients.

The third ODE is called **Bessel's equation**. If we put it in standard form

$$R'' + \frac{1}{r} R' + \left(n^2 - \frac{m^2}{r^2}\right) R = 0$$

we see that it has a regular singular point at $r = 0$.

The theory and application of Bessel functions (the solutions of Bessel type equations) is a very broad and important field. However, in this course, so that the main ideas of the generalized power series technique are presented as clearly as possible, we'll focus on the special case where $n = 1$ and $m = 0$;

$$r^2 R'' + r R' + r^2 R = 0 \tag{2}$$

Solutions of (2) are called *Bessel functions of order 0*.

Let us rewrite (2) as

$$x^2 y'' + x y' + x^2 y = 0 \tag{3}$$

(just changing the labels of variables to the way we usually write an ODE in this course). We shall look for solutions of (3) of the form

$$\begin{aligned}
y &= x^r \sum_{n=0}^{\infty} a_n x^n = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \cdots \\
&= \sum_{n=0}^{\infty} a_n x^{n+r}
\end{aligned}$$

We can assume (without loss of generality) that $a_0 \neq 0$; so that $a_0 x^r$ really is the leading term of this solution.

We have

$$\begin{aligned}
x^2 y'' &= x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} \\
x y' &= x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} \\
x^2 y &= x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r+2} = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}
\end{aligned}$$

and so when we replace $x^2 y''$, xy' and $x^2 y$ with their series expressions we get

$$0 = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

[1] By weakly coupled, I mean that the three ordinary differential equations are related to each other only by the fact that they involve the same constants m and n.

The first two series begin two steps before the last series, so before we can combine the power series we have to peel off the two initial terms of the first two series:

$$\begin{aligned}
0 &= (r)(r-1)a_0x^r + (r+1)(r)a_1x^{r+1} + \sum_{n=2}^{\infty}(n+r)(n+r-1)a_nx^{n+r} \\
&\quad ra_0x^r + (r+1)a_1x^{r+1} + \sum_{n=2}^{\infty}(n+r)a_nx^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r} \\
&= [r(r-1)+r]a_0x^r + (r(r+1)+r+1)a_1x^{r+1} \\
&\quad + \sum_{r=2}[(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2}]x^{n+r}
\end{aligned}$$

Note that we have now stratified the right hand side as sum of terms, each term having a distinct power of x as a factor.

We now demand that the total coefficient of each power of x separately vanish.

The lowest order term is

$$[r(r-1)+r]a_0x^r = r^2a_0x^r$$

For this to vanish for all x we need

$$r^2a_0 = 0 \quad \Rightarrow \quad r = 0$$

since our ansatz for y assumes that $a_0 \neq 0$.

The next higher order term is (using $r = 0$)

$$(r(r+1)+r+1)a_1x^{r+1} = (0(0+1)+0+1)a_1x = a_1x$$

So if this to vanish for all x we must have $a_1 = 0$.

Let's now look at the terms in the sum $\sum_{n=2}^{\infty}$. These are

$$\begin{aligned}
[(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2}]x^{n+r} &= [n(n-1)a_n + na_n + a_{n-2}]x^n \\
&= (na_n + a_{n-2})x^n
\end{aligned}$$

For these to all vanish we need

$$a_n = \frac{-a_{n-2}}{n} \quad , \quad n = 2,3,4,5,\dots$$

We can now begin to write down a solution. We have

$$\begin{aligned}
a_0 &= \text{arbitrary constant} \\
a_1 &= 0 \\
a_2 &= \frac{-a_0}{2} \\
a_3 &= \frac{-a_1}{3} = -\frac{0}{3} = 0 \\
a_4 &= \frac{-a_2}{4} = \frac{a_0}{4\cdot 2} = \frac{a_0}{2^2 2!} \\
a_5 &= -\frac{a_3}{2} = 0 \\
a_6 &= -\frac{a_4}{6} = -\frac{a_0}{6\cdot 4\cdot 2} = -\frac{a_0}{2^3 3!}
\end{aligned}$$

We thus observe the following pattern

$$a_n = \begin{cases} (-1)^k \frac{a_0}{2^k k!} & \text{if } n = 2k \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

And so we can write

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} x^{2k}$$

as the solution to (3).

2. Another example

Let's now consider the differential equation

$$2x^2y'' - xy' + (1+x)y = 0 \quad . \tag{28.1}$$

This equation evidently has a regular singular point at $x = 0$. We will look for a solution around $x = 0$ by making an ansatz for $y(x)$ by combining our ansatz for power series solutions about regular points with the ansatz we made for Euler type equations. More explicitly, we shall take

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \quad . \tag{28.2}$$

We can suppose without loss of generality that $a_0 \neq 0$; i.e., we assume r to be chosen such that the first nonzero term in the series is $a_o x^r$. Plugging (28.2) into (28.1) yields

(28.3)

$$\begin{aligned} 0 &= 2x^2 \textstyle\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2} - x\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1} + (1+x)\sum_{n=0}^{\infty} a_n x^{r+n} \\ &= \textstyle\sum_{n=0}^{\infty} 2(r+n)(r+n-1)a_n x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1} \\ &= \textstyle\sum_{n=0}^{\infty} \left(2(r+n)(r+n-1) - (r+n) + 1\right) a_n x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} \\ &= (2r)(r-1) - r + 1)\, a_0 + \textstyle\sum_{n=1}^{\infty} \left(\left(2(r+n)(r+n-1) - (r+n) + 1\right) a_n + a_{n-1}\right) x^{r+n} \end{aligned}$$

Hence, we need

$$0 = (2r)(r-1) - r + 1 = 2r^2 - 3r + 1 \tag{28.4}$$

$$0 = a_{n-1} + \left(2(r+n)(r+n-1) - (r+n) + 1\right) a_n \tag{28.5}$$

The first relation is a quadratic equation for r. It is called the **indicial equation** for (28.1). Since

$$2r^2 - 3r + 1 = (2r-1)(r-1) \tag{28.6}$$

we must have

$$r = \frac{1}{2}, 1 \tag{28.7}$$

The second equation (28.5) furnishes a recursion relation that allows us to fix all coefficients a_n in terms of a_o and r.

Setting $r = \frac{1}{2}$ we have

$$\begin{aligned} 0 &= a_{n-1} + \left(2(\tfrac{1}{2}+n)^2 - 3(\tfrac{1}{2}+n) + 1\right) a_n \\ &= a_{n-1} + [n(2n-1)]\, a_n \end{aligned} \tag{28.8}$$

so

$$a_n = \frac{-a_{n-1}}{n(2n-1)} \tag{28.9}$$

Thus,

$$\begin{aligned} a_1 &= \tfrac{-a_0}{(1)(2-1)} = -a_0 \\ a_2 &= \tfrac{-a_1}{(2)(4-1)} = \tfrac{a_0}{6} \\ a_3 &= \tfrac{-a_2}{(3)(6-1)} = \tfrac{-a_0}{90} \end{aligned} \tag{28.10}$$

So one solution would be

$$y_1(x) = a_0 x^{1/2}\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots\right) \quad . \tag{28.11}$$

When $r = 1$ we have

$$0 = a_{n-1} + \left(2(1+n)^2 - 3(1+n) + 1\right) a_n \tag{28.12}$$

or

$$a_n = \frac{-1}{2(1+n)^2 - 3(1+n) + 1} a_{n-1} = \frac{-a_{n-1}}{n(2n+1)} \quad . \tag{28.13}$$

So

$$\begin{aligned} a_1 &= \tfrac{-a_0}{1(2+1)} = -\tfrac{a_0}{3} \\ a_2 &= \tfrac{-a_1}{2(4+1)} = \tfrac{a_0}{30} \\ a_3 &= \tfrac{-a_2}{3(6+1)} = -\tfrac{a_0}{630} \end{aligned} \tag{28.14}$$

Thus, a second solution of (28.1) would be

$$y_2(x) = a_o x\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \cdots\right) \quad . \tag{28.15}$$

The general solution of (28.1) will be a linear combination of $y_1(x)$ and $y_2(x)$:

$$y(x) = c_1 x^{1/2}\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots\right) + c_2 x\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \cdots\right) \quad . \tag{28.16}$$

In summary, to find a solution of (28.1), we

(1) Assume there is a solution of the form $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$, with $a_0 \neq 0$.
(2) Plug this expression for $y(x)$ into the differential equation and set the total coefficients of each power of x equal to zero. This lead to
  (i) a quadratic equation for r (the indicial equation)
  (ii) a set of recursion relations relating the coefficients a_n
(3) Find the two roots r_1 and r_2 of the indicial equations, and then, for each root r_i used the recursion relations to express all the coefficients a_n in terms of a_o.
(4) Write down a corresponding solution for each root y_i(x) for each root r_i of the indicial equation.
(5) Write down the general solution as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad .$$

WARNING: This technique works produces two linearly independent solutions only when:

(i) **There are two distinct roots $\mathbf{r}_1$ and $\mathbf{r}_2$ of the indicial equation.**
(ii) **The difference $\mathbf{r}_1 - \mathbf{r}_2$ is not an integer.**

See Sections 5.7 and 5.8 of the text for a discussion of what happens and how to procede when these criteria are not meet.

Let's do another example

$$2xy'' + y' - y = 0$$

This equation has a regular singular point at $x = 0$. So we'll try to find a solution in the form of a generalized power series about $x = 0$.

Ansatz: $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$\begin{aligned}
0 &= 2xy'' + y' - y \\
&= \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=0}^{\infty} [(n+r)(2n+2r-2) a_n + (n+r) a_n] x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=0}^{\infty} [(n+r)(2n+2r-1) a_n] x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=-1}^{\infty} [(n+r+1)(2n+2r+1) a_{n+1}] x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= r(2r-1) a_0 x^{r-1} + \sum_{n=0}^{\infty} [(n+r+1)(2n+2r+1) a_{n+1}] x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= r(2r-1) a_0 x^{r-1} + \sum_{n=0}^{\infty} [(n+r+1)(2n+2r+1) a_{n+1} - a_n] x^{n+r}
\end{aligned}$$

We now demand that the total coefficient of each distinct power of x vanish. This leads us to the following equations

$$\begin{aligned}
r(2r-1) a_0 &= 0 \\
(n+r+1)(2n+2r+1) a_{n+1} - a_n &= 0 \qquad , \qquad n = 0, 1, 2, 3, \ldots
\end{aligned}$$

We always assume that $a_0 \neq 0$ (otherwise the leading term of our ansatz for y is not $a_0 x^r$). Hence, the first equation requires

$$r(2r-1) = 0 \quad \Rightarrow \quad r = 0, \frac{1}{2}$$

We thus have determined that there are two and only two possible choices for r. The coefficients a_n will be determined by

$$a_{n+1} = \frac{a_n}{(n+r+1)(2n+2r+1)} \qquad , \qquad n = 0, 1, 2, 3, \ldots$$

- Solution with $r = 0$

  The recursion relations in this case reduce to

$$a_{n+1} = \frac{a_n}{(n+1)(2n+1)} \qquad , \qquad n = 0, 1, 2, 3, \ldots$$

  Thus, if $a_0 = c_1$, then

$$\begin{aligned}
a_1 &= \frac{a_0}{(1)(1)} = c_1 \\
a_2 &= \frac{a_1}{(2)(3)} = \frac{c_1}{6} \\
a_3 &= \frac{a_2}{(3)(5)} = \frac{c_1}{90}
\end{aligned}$$

  and so the first four terms of this solution will be

$$\begin{aligned}
y &= a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \cdots \\
&= c_1 + c_1 x + \frac{c_1}{6} x^2 + \frac{c_1}{90} x^3 + \cdots \\
&= c_1 y_1
\end{aligned}$$

where

$$y_1 = 1 + x + \frac{1}{2}x^2 + \frac{1}{90}x^3 + \cdots$$

- Solution with $r = \frac{1}{2}$

In this case the recursion relations reduce to

$$a_{n+1} = \frac{a_n}{(n+r+1)(2n+2r+1)} = \frac{a_n}{\left(n+\frac{3}{2}\right)(2n+2)} = \frac{a_n}{(2n+3)(n+1)} \quad , \quad n = 0,1,2,3,\dots$$

Setting $a_0 = c_2$ we then get

$$\begin{aligned} a_1 &= \frac{a_0}{(3)(1)} = \frac{c_2}{3} \\ a_2 &= \frac{a_1}{(5)(2)} = \frac{c_2}{30} \\ a_3 &= \frac{a_2}{(7)(3)} = \frac{c_2}{630} \end{aligned}$$

and so the first four terms of the solution will be

$$\begin{aligned} y &= a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \cdots \\ &= c_2 x^{\frac{1}{2}} + \frac{c_2}{3} x^{3/2} + \frac{c_2}{30} x^{5/2} + \frac{c_2}{30} x^{7/2} + \cdots \\ &= c_2 \sqrt{x}\left(1 + \frac{1}{3}x + \frac{1}{30}x^2 + \frac{1}{630}x^3 + \cdots\right) \\ &= c_2 y_2 \end{aligned}$$

where

$$y_2(x) = \sqrt{x}\left(1 + \frac{1}{3}x + \frac{1}{30}x^2 + \frac{1}{630}x^3 + \cdots\right)$$

3. The case when $r_1 - r_2$ is an integer

As we remarked above, when the indicial equation has two distinct roots, r_1 and r_2, and $r_1 - r_2$ is not an integer then the generalized power series method will lead to two independent solutions

$$\begin{aligned} y_1 &= x^{r_1} \sum_{n=0}^{\infty} a_n x^n \\ y_2 &= x^{r_2} \sum_{n=0}^{\infty} a_n' x^n \end{aligned}$$

where the coefficients a_n, a_n' are determined by applying the recursion relations using, respectively, $r = r_1$ and $r = r_2$ in the recursive formulae.

If on the other hand, $r_1 - r_2$ is an integer, it turns out that only one of the roots (the larger one) can be employed to solve the recursion relations (typically, if one tries to use the smaller root to compute the a_n, then one encounters a division by zero in the recursive computation of the a_n).

Nevertheless, there are techniques that yield a second solution when $r_1 - r_2$ is an integer. However, these techniques are beyond the scope of this course. The interested student can read up on this situation in Section 5.6 of the text.