LECTURE 27

Singular Points and the Convergence of Series Solutions

As it stands our method of finding power series solutions to differential equations of the form

$$y'' + p(x)y' + q(x)y = 0 \tag{27.1}$$

is purely formal. For a series solution

$$\sum_{n=0}^{\infty} a_n (x - x_o)^n \tag{27.2}$$

might not converge for any finite x (and we need the series to converge if we are to use it to define a legitimate function of x).

To discuss this situation with the care it deserves, we must first introduce a little more formal development.

DEFINITION 27.1. *A function f is said to be **analytic** about the point x_o if it has a power (Taylor) series expansion about that point;*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n \tag{27.3}$$

with some non-zero radius of convergence.

THEOREM 27.2. *If the functions $p(x)$ and $q(x)$ are analytic at the point x_o, then one can find (linear) functions a_n of a_o and a_1 so that the general solution of*

$$y'' + p(x)y' + q(x)y = 0 \tag{27.4}$$

can be expressed as a power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n(a_o, a_1)(x - x_o)^n = a_o y_1(x) + a_1 y_2(x) \quad , \tag{27.5}$$

where y_1 and y_2 are two linearly independent solutions of (27.4) which are analytic at x_o. Moreover, the radius of convergence of the power series expansions of y_1 and y_2 is at least as large as the minimum of the radii of convergence of the power series for $p(x)$ and $q(x)$.

Thus, if we know the radii of convergence $p(x)$ and $q(x)$ we needn't do anything as laborious as compute the radius of convergence of our solution using the ratio test (which is really only going to work if you have an explicit formula for the ratio $\left|\frac{a_n}{a_{n+1}}\right|$). The following theorem is very useful in determining the radii of convergence of the power series expansions of $p(x)$ and $q(x)$.

THEOREM 27.3. *If $f(x)$ is the ratio of two polynomial functions;*

$$f(x) = \frac{A(x)}{B(x)} \tag{27.6}$$

and $B(x_o) \neq 0$, then

(i) *$f(x)$ has a power series expansion about $x = x_o$.*
(ii) *The radius of convergence of this power series about x_o is equal to the distance (in the complex plane) between x_o and the nearest zero of $B(x)$.*

EXAMPLE 27.4. What is the radius of convergence of the Taylor expansion of

$$f(x) = \frac{1}{1+x^2} \tag{27.7}$$

about $x = 0$? About $x = 2$?

The denominator vanishes when $x = \pm i$. To determine the radius of convergence we need only compute the distance in the complex plane between $x = \pm i$ and the expansion point in question. In terms of the Cartesian coordinates of the complex plane the points $x = \pm i$ are given by, respectively, (0,1) and (0,-1), while the real points 0 and 2 are given by, respectively, (0,0) and (2,0). Thus, the distance between 0 and $\pm i$ is

$$\sqrt{(0-0)^2 + (0 \mp 1)^2} = 1 \quad , \tag{27.8}$$

so the radius of convergence of the Taylor series expansion of $f(x)$ about 0 is 1. The distance between 2 and $\pm i$ is

$$\sqrt{(2-0)^2 + (0 \mp 1)^2} = \sqrt{5} \quad , \tag{27.9}$$

so the radius of convergence of the Taylor series expansion of $f(x)$ about $x = 2$ is $\sqrt{5}$.

EXAMPLE 27.5. Find the radius of convergence of the Taylor series expansion of

$$f(x) = \frac{1}{(x+2)(x-3)} \tag{27.10}$$

about $x_o = 4$.

The zeros of the denominator are $x = -2, 3$. The distance (in the complex plane from $x_o = 4 = (4,0)$ to the closest zero $x = 3 = (3,0)$ is

$$\sqrt{(4-3)^2 - (0-0)^2} = 1 \quad , \tag{27.11}$$

so the radius convergence of the Taylor series expansion of $f(x)$ about $x_o = 4$ is 1.

Let us now combine the two theorems to determine the minimal radius of convergence of the power series solution of

$$(x^2 - 2x - 3)y'' + xy' + 4(x-3)y = 0 \tag{27.12}$$

about $x_o = 4$.

This differential equation is equivalent to

$$y'' + \frac{x}{x^2 - 2x - 3}y' + \frac{4}{x+2}y = 0 \quad . \tag{27.13}$$

The zeros of $x^2 - 2x - 3 = (x-3)(x+2)$ are $x = 3, -2$, and -2 is the only zero $x + 2$. Therefore the singularity of

$$p(x) = \frac{x}{x^2 - 2x - 3} \tag{27.14}$$

and/or

$$q(x) = \frac{4}{x+2} \tag{27.15}$$

that is closest to $x_o = 4$ is $x = 4$. Since $|4 - 3| = 1$, the radius of convergence of a power series expansion of $p(x)$ about $x_o = 4$ is 1, the minimal radius of convergence of a series solution of (27.12) will be 1.

1. Solutions near Regular Singular Points

Recall from the preceding section that a differential equation

$$y'' + p(x)y' + q(x) = 0 \tag{27.16}$$

always has a power series solution about a point x_o so long as the functions $p(x)$ and $q(x)$ have power series expansions around x_o. We will now discuss the case when $p(x)$ or $q(x)$ has a singularity at x_o.

Example 27.6. Consider the differential equation

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0 \quad . \tag{27.17}$$

One easily verifies that the functions

$$\begin{array}{rcl} y_1(x) & = & x \\ y_2(x) & = & x^2 \end{array} \tag{27.18}$$

form a linearly independent set of solutions to (27.17). Note that $y_1(x)$ and $y_2(x)$ are both well-behaved functions at the point $x = 0$ (where $p(x) = -\frac{2}{x}$ and $q(x) = \frac{2}{x^2}$ both have a singularity).

Example 27.7. Consider the following differential equation:

$$y'' - \frac{2}{x^2}y = 0 \quad . \tag{27.19}$$

Because of the singularity of $\frac{-2}{x^2}$ at $x_o = 0$, Theorem 27.2 above does not guarantee the existence of a power series solution around $x_o = 0$. However, one can easily check that

$$\begin{array}{rcl} y_1(x) & = & x^2 \\ y_2(x) & = & \frac{1}{x} \end{array} \tag{27.20}$$

form a set of linearly independent solutions to (27.19). We note that the solution $y_1(x)$ is a perfectly well-behaved at the point $x = 0$; however, the solution $y_2(x)$ is singular at the point $x = 0$.

The preceding examples show that just because a differential equation has a singularity it does not necessarily follow that there are no solutions or even that the solutions are ill-behaved at the singularity.

Definition 27.8. *A differential equation of the form*

$$y'' + p(x)y' + q(x)y = 0 \tag{27.21}$$

is said to have a ***singular point*** *at x_o if either*

$$\lim_{x \to x_o} p(x) \tag{27.22}$$

or

$$\lim_{x \to x_o} q(x) \tag{27.23}$$

does not exist.

Definition 27.9. *Suppose x_o is a singular point of*

$$y'' + p(x)y' + q(x)y = 0 \quad . \tag{27.24}$$

x_o is said to be a ***regular singular point*** *(of this differential equation) if the singularity of $p(x)$ is no worse than*

$$\frac{1}{x - x_o} \tag{27.25}$$

and the singularity of $q(x)$ is no worse than

$$\frac{1}{(x - x_0)^2} \quad . \tag{27.26}$$

More precisely, x_o is a regular singular point if both

$$\lim_{x \to x_o} (x - x_o)\, p(x) \tag{27.27}$$

and

$$\lim_{x\to x_o} (x-x_o)^2\, q(x) \tag{27.28}$$

exist. Otherwise, x_o is called an ***irregular singular point****.*

Example 27.10. The differential equation

$$y'' + \frac{3}{(x-1)(x+1)^2}y' + \frac{2x+1}{(x-2)^2(x+2)(x-1)^3}y = 0 \tag{27.29}$$

has singular points at $x = 1, -1, 2, -2$.

Now

Singular Point x_o	$\lim_{x\to x_o}(x-x_o)p(x)$	$\lim_{x\to x_o}(x-x_o)^2q(x)$	Type
1	$\frac{3}{4}$	∞	irregular
-1	∞	0	irregular
2	0	$\frac{5}{4}$	regular
-2	0	0	regular

So $x = \pm 1$ are irregular singular points and $x = \pm 2$ are regular singular points.

Example 27.11. Identify and classify the singular points of

$$x^2(1-x^2)^2y'' + x(1+x)^2y' + (1-x)y' \quad . \tag{27.30}$$

In this case, when we divide by $x^2(1-x^2)^2$ to put the equation in standard form, we have

$$p(x) = \frac{x(1+x)(1+x)}{x^2(1+x)^2(1-x)^2} = \frac{1}{x(1-x)^2} \tag{27.31}$$

and

$$q(x) = \frac{(1-x)}{x^2(1+x)^2(1-x)^2} = \frac{1}{x^2(1+x)^2(1-x)} \quad . \tag{27.32}$$

Thus, we have regular singular points at $x = 0, -1$ and an irregular singular point at $x = 1$.