LECTURE 26

Differential Equations with Polynomial Coefficients

In the last lecture we considered a number of examples of differential equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{26.1}$$

and looked for solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n \quad . \tag{26.2}$$

Before considering one more example, let me first articulate the general procedure.

Step 1. Substitute (26.2) into (26.1). This will produce an equation of the form

$$\begin{aligned} 0 \quad = \quad & \textstyle\sum_{n=0}^{\infty} n(n-1)a_n P(x) (x - x_o)^{n-2} + \sum_{n=0}^{\infty} n a_n Q(x) (x - x_o)^{n-1} \\ & + \textstyle\sum_{n=0}^{\infty} a_n R(x) (x-1)^n \end{aligned} \tag{26.3}$$

Step 2. Unfortunately, depending on the nature of the polynomials, it may happen that none of three series in (26.3) is a power series in $(x - x_o)$. For example, if $P(x) = x^2$ and $x_o = 1$, then the first series is

$$\sum_{n=0}^{\infty} n(n-1)a_n x^2 (x-1)^2 \tag{26.4}$$

which is not a power series (i.e., an expression of the form $\sum b_n (x-1)^n$ with each b_n a constant). To convert the series in (26.3) into to power series we must replace the polynomials $P(x)$, $Q(x)$, and $R(x)$ with their Taylor expansions about $x_o = 1$. If we set

$$\begin{aligned} p_n &= \tfrac{1}{n!} \tfrac{d^n P}{dx^n} (x_o) \\ q_n &= \tfrac{1}{n!} \tfrac{d^n Q}{dx^n} (x_o) \\ r_n &= \tfrac{1}{n!} \tfrac{d^n R}{dx^n} (x_o) \end{aligned} \tag{26.5}$$

we can write

$$\begin{aligned} P(x) &= \textstyle\sum_{i=0}^{\infty} p_n (x - x_o)^n \quad , \\ Q(x) &= \textstyle\sum_{i=0}^{\infty} q_n (x - x_o)^n \quad , \\ R(x) &= \textstyle\sum_{i=0}^{\infty} r_n (x - x_o)^n \quad . \end{aligned} \tag{26.6}$$

Actually, since polynomial of degree D can have at most D non-vanishing derivatives, each of the Taylor expansions (26.6) will terminate after a finite number of terms:

$$\begin{aligned} P(x) &= \textstyle\sum_{i=0}^{d_P} p_n (x - x_o)^n \quad , \\ Q(x) &= \textstyle\sum_{i=0}^{d_Q} q_n (x - x_o)^n \quad , \\ R(x) &= \textstyle\sum_{i=0}^{d_R} r_n (x - x_o)^n \quad . \end{aligned} \tag{26.7}$$

where d_P, d_Q, and d_R are the degrees of the polynomials $P(x)$, $Q(x)$, and $R(X)$. Inserting the expression (26.7) into (26.3) we get

$$\begin{aligned} 0 &= \textstyle\sum_{n=0}^{\infty}\sum_{i=0}^{d_P} n(n-1)a_n p_n \left(x-x_o\right)^{n+i-2} \\ &\quad + \textstyle\sum_{n=0}^{\infty}\sum_{i=0}^{d_P} n a_n q_n \left(x-x_o\right)^{n+i-1} \\ &\quad + \textstyle\sum_{n=0}^{\infty}\sum_{i=0}^{d_R} a_n r_n \left(x-x_o\right)^{n+i} \\ &= \textstyle\sum_{i=0}^{d_P}\sum_{n=0}^{\infty} n(n-1)a_n p_n \left(x-x_o\right)^{n+i-2} \\ &\quad + \textstyle\sum_{i=0}^{d_Q}\sum_{n=0}^{\infty} n a_n q_n \left(x-x_o\right)^{n+i-1} \\ &\quad + \textstyle\sum_{i=0}^{d_R}\sum_{n=0}^{\infty} a_n r_n \left(x-x_o\right)^{n+i} \end{aligned} \tag{26.8}$$

or

$$\begin{aligned} 0 &= \textstyle\sum_{n=0}^{\infty} n(n-1)p_0 a_n \left(x-x_o\right)^{n-2} + \cdots + \sum_{n=0}^{\infty} n(n-1)p_{d_P} a_n \left(x-x_o\right)^{n+d_P-2} \\ &\quad + \textstyle\sum_{n=0}^{\infty} n q_0 a_n \left(x-x_o\right)^{n-1} + \cdots + \sum_{n=0}^{\infty} n q_{d_Q} a_n \left(x-x_o\right)^{n+d_Q-1} \\ &\quad + \textstyle\sum_{n=0}^{\infty} r_0 a_n \left(x-x_o\right)^{n} + \cdots + \sum_{n=0}^{\infty} r_{d_R} a_n \left(x-x_o\right)^{n+d_R} \end{aligned} \tag{26.9}$$

Step 3. The next step is to collect all the terms consisting of like factors of $\left(x-x_o\right)^i$. To accomplish this we shift the summation index n in each series in (26.9) so that the k^{th} term in the new series has $\left(x-x_o\right)^k$ as a factor. One obtains

(26.10)

$$\begin{aligned} 0 &= \textstyle\sum_{k=-2}^{\infty}(k+2)(k+1)p_0 a_{k+2} \left(x-x_o\right)^k + \cdots \\ &\quad \cdots + \textstyle\sum_{n=-2+d_P}^{\infty}(k+2-d_P)(k+1-d_P)p_{d_P} a_{k+2-d_P} \left(x-x_o\right)^k \\ &\quad + \textstyle\sum_{k=0}^{\infty}(k+1)q_0 a_{k+1} \left(x-x_o\right)^k + \cdots + \sum_{k=-1+d_Q}^{\infty} \left(k+1-d_Q\right) q_{d_Q} a_{k+1-d_Q} \left(x-x_o\right)^k \\ &\quad + \textstyle\sum_{k=0}^{\infty} r_0 a_k \left(x-x_o\right)^k + \cdots + \sum_{k=d_R}^{\infty} r_{d_R} a_{k-d_R} \left(x-x_o\right)^k \end{aligned}$$

Here one must be a bit careful. Notice that the various series appearing in the above equation **do not** have the same initial value of k. Before consolidating the various series in (26.10) in a single series we must make sure they all start off at the same value of k. I will discuss this point momentarily with an example. But certainly for k large enough all the series in (26.10) will contribute terms proportional to $\left(x-x_o\right)^k$. One can then read off from (26.10) the general recursion relation

$$\begin{aligned} 0 &= (k+2)(k+1)p_o a_{k+2} + \cdots (k+2-d_P)(k+1-d_P)a_{k+2-d_P} \\ &\quad +(k+1)q_0 a_{k+1} + \cdots + (k+1-d_Q)q_{d_Q} a_{k+1-d_Q} \\ &\quad + r_0 a_k + \cdots + r_{d_R} a_{k-d_R} \end{aligned} \tag{26.11}$$

which is valid for $k > Max\left\{-2+d_P, -1+d_Q, d_R\right\}$. Actually, we can use this relation for all k so long as we consistently define

$$a_i = 0 \quad , \quad \text{if} i < 0. \tag{26.12}$$

Step 4. Use the recursion relation (26.11) to express all the coefficients a_n in terms of a_0 and a_1 (you may also need to use the relations $0 = a_{-1} = a_{-2} = a_{-3} \cdots$ coming from (26.12)).

EXAMPLE 26.1. Find a power series solution of

$$x^2 y'' + (x+1)y = 0 \tag{26.13}$$

about the point $x_o = 1$.

Plugging

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n \tag{26.14}$$

into (26.13) yields

$$0=\sum_{n=0}^{\infty}n(n-1)a_nx^2(x-1)^{n-2}+\sum_{n=0}^{\infty}a_n(x+1)(x-1)^n \quad . \tag{26.15}$$

Now the Taylor expansions of $f(x)=x^2$ and $g(x)=x+1$ about $x_o=1$ are

$$\begin{aligned} x^2 &= 1+2(x-1)+(x-1)^2 \\ x+1 &= 2+(x-1) \quad . \end{aligned} \tag{26.16}$$

Plugging the right hand sides of (26.16) into (26.15) yields

$$\begin{aligned} 0 &= \textstyle\sum_{n=0}^{\infty} n(n-1)a_n\left(1+2(x-1)+(x-1)^2\right)(x-1)^{n-2} \\ &\quad +\textstyle\sum_{n=0}^{\infty} a_n\left(2+(x-1)\right)(x-1)^n \\ &= \textstyle\sum_{n=0}^{\infty} n(n-1)a_n(x-1)^{n-2}+\sum_{n=0}^{\infty} 2n(n-1)a_n(x-1)^{n-1} \\ &\quad +\textstyle\sum_{n=0}^{\infty} n(n-1)a_n(x-1)^n+\sum_{n=0}^{\infty} 2a_n(x-1)^n+\sum_{n=0}^{\infty} a_n(x-1)^{n+1} \end{aligned} \tag{26.17}$$

We now shift the summation indices in each series so that in the k^{th} term, $(x-1)$ appears to the k^{th} power. One gets

$$\begin{aligned} 0 &= 0+0+\textstyle\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-1)^k \quad +0+\sum_{k=0}^{\infty}2(k+1)k(x-1)^ka_{k+1}(x-1)^k \\ &\quad +\textstyle\sum_{k=0}^{\infty}k(k-1)a_k(x-1)^k+\sum_{k=0}^{\infty}2a_k(x-1)^k+\sum_{k=1}^{\infty}a_{k-1}(x-1)^k \end{aligned} \tag{26.18}$$

Unfortunately, the last series begins with $k=1$, instead of $k=0$. This, however, is easy to remedy; we simply $a_{-1}=0$, so that

$$\sum_{k=0}^{\infty}a_{k-1}(x-1)^k=0(x-k)^{-1}+\sum_{k=1}^{\infty}a_{k-1}(x-1)^k=\sum_{k=1}^{\infty}a_{k-1}(x-1)^k \quad . \tag{26.19}$$

Thus, having arranged things so that all series start off at the same point $k=0$ and we now consolidate the right hand side of (26.18) into a single series:

$$\begin{aligned} 0 &= \textstyle\sum_{k=0}^{\infty}\left((k+2)(k+1)a_{k+2}+2(k+1)ka_{k+1}+k(k-1)a_k+2a_k+a_{k-1}\right)(x-1)^k \\ &= \textstyle\sum_{k=0}^{\infty}\left((k+2)(k+1)a_{k+2}+2k(k+1)a_{k+1}+(k^2-k+2)a_k+a_{k-1}\right)(x-1)^k \end{aligned} \tag{26.20}$$

The demand that the total coefficient of $(x-1)^k$ vanish then implies

$$a_{k+2}=\frac{-2k(k+1)a_{k+1}-(k^2-k+2)a_k-a_{k-1}}{(k+2)(k+1)} \quad . \tag{26.21}$$

Thus, given that $a_{-1}=0$, we have

$$\begin{aligned} a_2 &= \tfrac{0-2a_0-0}{(26.2)(26.1)}=-a_0 \\ a_3 &= \tfrac{(-2)(26.3)a_2-(26.2)a_1-a_0}{(26.3)(26.2)}=\tfrac{-7a_0-2a_1}{6} \\ a_4 &= \tfrac{(-4)(26.3)a_3-4a_2-a_1}{(26.4)(26.3)}=\tfrac{(14a_0-4a_1+4a_0-a_1)}{12}=\tfrac{18a_0-5a_1}{12} \end{aligned} \tag{26.22}$$

Thus, to the order of $(x-1)^4$ the general solution of (26.13) is

$$\begin{aligned} y(x) &= a_0+a_1(x-1)-a_0(x-1)^2-\tfrac{7a_0-2a_1}{6}(x-1)^3 \\ &\quad +\tfrac{18a_0-5a_1}{12}(x-1)^4+\cdots \\ &= a_0\left(1-(x-1)^2-\tfrac{7}{6}(x-1)^3+\tfrac{3}{2}(x-1)^4+\cdots\right) \\ &\quad +a_1\left((x-1)+\tfrac{1}{3}(x-1)^3-\tfrac{5}{12}(x-1)^4+\cdots\right) \end{aligned} \tag{26.23}$$