Math 2233.003
SECOND EXAM
1:30 – 2:20 am, April 10, 2001

Name:______________________________

1. Given that $y_1(x) = x$ and $y_2(x) = x^3$ are solutions to $x^2y'' - 3xy' + 3y = 0$

(a) (5 pts) Show that the functions $y_1(x)$ and $y_2(x)$ are linearly independent.

- $$W[y_1, y_2] = (x)(x^3)' - (x)'(x^3) = (x)(3x^2) - (1)(x^3) = 2x^3$$

∎

(b) (5 pts) Write down the general solution.

- $$y = c_1y_1 + c_2y_2 = c_1x + c_2x^3$$

∎

(c) (5 pts) Find the solution satisfying the initial conditions $y(1) = 2$, $y'(1) = 1$.

- $$\left.\begin{array}{r} 2 = y(1) = c_1 + c_2 \\ 1 = y'(1) = c_1 + 3c_2 \end{array}\right\} \quad \Rightarrow \quad c_1 = \frac{5}{2} \quad , \quad c_2 = -\frac{1}{2} \qquad \Rightarrow \quad y = \frac{5}{2}x - \frac{1}{2}x^3$$

∎

2. (10 pts) Given that $y_1(x) = x^{-2}$ is one solution of $x^2y'' + 6xy' + 6y = 0$, use Reduction of Order to determine the general solution.

- $$\begin{aligned} y_2 &= y_1 \int \frac{1}{[y_1]^2} \exp\left(-\int p(x)dx\right) dx = x^{-2} \int \frac{1}{x^{-4}} \exp\left(-\int \frac{6}{x}dx\right) dx = x^{-2} \int x^4 \exp\left(-6\ln|x|\right) dx \\ &= x^{-2} \int x^4 \left(x^{-6}\right) dx = x^{-2} \int x^{-2}dx = x^{-2}\left(-x^{-1}\right) = x^{-3} \end{aligned}$$

$$y(x) = c_1y_1 + c_2y_2 = c_1x^{-2} + c_2x^{-3}$$

∎

3. (10 pts) Explain in words and formulas how one could use Reduction of Order and the Method of Variation of Parameters to construct the general solution of $x^2y'' - 2y = 3x^2 - 1$, given that $y_1(x) = x^2$ is a solution of $x^2y'' - 2y = 0$. (It is not necessary to actually carry out any of the calculations.)

- Use Reduction of Order Formula to calculate a second linearly independent solution of the homogeneous equation (using $y_1 = x^2$ and $p(x) = 0$)
$$y_2 = y_1 \int \frac{1}{[y_1]^2} \exp\left(-\int pdx\right) dx \qquad \text{(Reduction of Order Formula)}$$
- Calculate a particular solution of the inhomogeneous equation (using y_2 determined above and $g = \frac{3x^2-1}{x^2}$).
$$y_p = -y_1 \int \frac{y_2g}{y_1y_2' - y_1'y_2}dx + y_{21} \int \frac{y_1g}{y_1y_2' - y_1'y_2}dx \qquad \text{(Variation of Parameters Formula)}$$
- General solution will be
$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x)$$

∎

4. Determine the general solution of the following differential equations.

(a) (5 pts) $y'' - 4y' - y = 0$

- This is second order linear with constant coefficients. The roots of the characteristic polynomial are

$$0 = \lambda^2 - 4\lambda - y \quad \Rightarrow \quad \lambda = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$$

(two distinct real roots) and so

$$y(x) = c_1 e^{\left(2+\sqrt{5}\right)x} + c_2 e^{\left(2-\sqrt{5}\right)x}$$

∎

(b) (5 pts) $y'' + 12y' + 36y = 0$

- This is second order linear with constant coefficients. The roots of the characteristic polynomial are

$$0 = \lambda^2 + 12\lambda + 36 = (\lambda+6)^2 \quad \Rightarrow \quad \lambda = -6$$

(a single real root) and so

$$y = c_1 e^{-6x} + c_2 x e^{-6x}$$

∎

(c) (5 pts) $y'' - 4y' + 13y = 0$

- This is second order linear with constant coefficients. The roots of the characteristic polynomial are

$$0 = \lambda^2 - 4\lambda + 13 \quad \Rightarrow \quad \lambda = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

(a pair of complex conjugate roots) and so

$$y = c_1 e^{2x}\cos(3x) + c_2 e^{2x}\sin(3x)$$

∎

5. Given that $y_1(x) = e^{2x}$ and $y_2(x) = e^{-3x}$ are solutions of $y'' + y' - 6y = 0$.

(a) (10 pts) Use the Method of Variation of Parameters to find a particular solution of $y'' + y' - 6y = e^x$.

-

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2 g}{y_1 y_2' - y_1' y_2} dx + y_2 \int \frac{y_1 g}{y_1 y_2' - y_1' y_2} dx = -e^{2x} \int \frac{\left(e^{-3x}\right) e^x}{-5e^{-x}} dx + e^{-3x} \int \frac{\left(e^{2x}\right) e^x}{-5e^{-x}} dx \\ &= \frac{1}{5} e^{2x} \int e^{-x} dx - \frac{1}{5} e^{-3x} \int e^{4x} dx = \frac{1}{5} e^{2x}\left(-e^{-x}\right) - \frac{1}{5} e^{-3x}\left(\frac{1}{4} e^{4x}\right) = -\left(\frac{1}{5} + \frac{1}{20}\right) e^x = -\frac{1}{4} e^x \end{aligned}$$

∎

(b) (5 pts) Find the solution satisfying $y(0) = 0$, $y'(0) = 2$.

- The general solution is $y = y_p + c_1 y_1 + c_2 y_2 = \frac{1}{4} e^x + c_1 e^{2x} + c_2 e^{-3x}$

$$\left.\begin{array}{r} 0 = -\frac{1}{4} + c_1 + c_2 \\ 2 = -\frac{1}{4} + 2c_1 - 3c_2 \end{array}\right\} \quad \Rightarrow \quad c_1 = \frac{3}{5} \quad , \quad c_2 = -\frac{7}{20} \qquad \Rightarrow \quad y = -\frac{1}{4} e^x + \frac{3}{5} e^{2x} - \frac{7}{20} e^{-3x}$$

∎

6. Find the general solution of the following Euler-type differential equations.

(a) (5 pts) $x^2y'' + 4xy' - 10y = 0$

- This is an Euler type equation:

$$0 = r(r-1) + 4r - 10 = r^2 + 3r - 10 \quad \Rightarrow \quad r = \frac{-3 \pm \sqrt{9+40}}{2} = \frac{-3 \pm 7}{2} = 2, -5$$

  The indicial equation has two real roots; and so

$$y = c_1x^2 + c_2x^{-5}$$

∎

(b) (5 pts) $x^2y'' - 5xy' + 9y = 0$

- This is an Euler type equation:

$$0 = r(r-1) - 5r + 9 = r^2 - 6r + 9 = (r-3)^2 \quad \Rightarrow \quad r = 3$$

  The indicial equation has a single real root; and so

$$y = c_1x^3 + c_2x^3 \ln|x|$$

∎

(c) (5 pts) $x^2y'' - 5xy' + 13y = 0$

- This is an Euler type equation:

$$0 = r(r-1) - 5r + 13 = r^2 - 6r + 13 \quad \Rightarrow \quad r = \frac{-6 \pm \sqrt{36-52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i$$

  The indicial equation has a pair of complex conjugate roots; and so

$$y = c_1x^{-3}\cos(2\ln|x|) + c_2x^{-3}\sin(2\ln|x|)$$

∎

7. Determine the general solution of the following differential equations.

(a) (10 pts) $y''' - y'' - 8y' + 12y = 0$ (Hint: $x+3$ is a factor of the characteristic polynomial for this equation.)

- Using polynomial division, one sees that the characteristic polynomial $\lambda^3 - \lambda^2 - 8\lambda + 12$ factorizes as

$$\lambda^3 - \lambda^2 - 8\lambda + 12 = (\lambda+3)\left(\lambda^2 - 4\lambda + 4\right) = (\lambda+3)(\lambda-2)^2$$

  and so we have a real root $\lambda = -3$ with multiplicity 1 and a real root $\lambda = 2$ with multiplicity $\dot{2}$. The general solution is thus

$$y = c_1e^{-3x} + c_2e^{2x} + c_3xe^{2x}$$

∎

(b) (10 pts) $y'''' + 6y'' + 9y = 0$ (Hint: the characteristic polynomial is a perfect square)

- The characteristic equation is

$$0 = \lambda^4 + 6\lambda^2 + 9 = \left(\lambda^2+3\right)^2 = \left[\left(\lambda+\sqrt{3}i\right)\left(\lambda-\sqrt{3}i\right)\right]^2 = \left(\lambda+\sqrt{3}i\right)^2\left(\lambda-\sqrt{3}i\right)^2$$

  and so we have a pair of complex roots with multiplicity two. The general solution is thus

$$\begin{aligned} y &= c_1e^{0x}\cos\left(\sqrt{3}x\right) + c_2e^{0x}\sin\left(\sqrt{3}x\right) + c_3e^{0x}x\cos\left(\sqrt{3}x\right) + c_4e^{0x}x\sin\left(\sqrt{3}x\right) \\ &= c_1\cos\left(\sqrt{3}x\right) + c_2\sin\left(\sqrt{3}x\right) + c_3x\cos\left(\sqrt{3}x\right) + c_4x\sin\left(\sqrt{3}x\right) \end{aligned}$$

∎