LECTURE 21

Higher Order Linear Equations

We now turn to the problem of constructing solutions of n^{th} order linear differential equations; i.e., differential equations of the form

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = g(x) \quad . \tag{21.1}$$

THEOREM 21.1. *If the functions $p_1(x), p_2(x), \ldots, p_n(x)$ and $g(x)$ are continuous and differentiable on an open interval $\alpha < x < \beta$, then there exists one and only one function y(x) satisfying (21.1) on the interval $\alpha < x < \beta$ and the initial conditions*

$$\begin{aligned} y(x_o) &= y_o \\ \tfrac{dy}{dx}(x_o) &= y'_o \\ &\vdots \\ \tfrac{d^{n-1}y}{dx^{n-1}}(x_o) &= y^{(n-1)} \quad . \end{aligned} \tag{21.2}$$

Recall that if y_1 and y_2 were solutions of a second order linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \quad . \tag{21.3}$$

and

$$W[y_1, y_2](x) = y_1(x)y'_2(X) - y'_1(x)y_2(x) \neq 0 \tag{21.4}$$

then every solution (21.3) can be expressed as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{21.5}$$

for some choice of constants c_1 and c_2.

The situation for n^{th} order linear equations is similar; however the explicit expression for the corresponding Wronskian is a bit tedious to write down for large n.

DEFINITION 21.2. *A set of functions $\{\phi_1, \phi_2, \ldots, \phi_n\}$ is said to be a linearly independent set on the interval $I = (\alpha, \beta)$ if there exists no choice of constants $c_1, \ldots, c_n$ such that*

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0 \qquad , \qquad \forall x \in I \tag{21.6}$$

except $c_1 = c_2 = \cdots = c_n = 0$.

THEOREM 21.3. *A set of (differentiable) functions $\{\phi_1, \phi_2, \ldots, \phi_n\}$ is linearly independent on an interval I if and only if*

$$0 \neq W[\phi_1, \ldots, \phi_n](x) \equiv Det \begin{pmatrix} \phi_1(x) & \phi_2(x) & \cdots & \phi_n(x) \\ \phi'_1(x) & \phi'_2(x) & \cdots & \phi'_n(x) \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)}(x) & \phi_2^{(n-1)}(x) & \cdots & \phi_n^{(n-1)}(x) \end{pmatrix} \tag{21.7}$$

on I.

Example: Three differentiable functions $f(x), g(x), h(x)$ are linearly independent if and only if

$$\begin{aligned} 0 \neq W[f,g,h](x) &= f(x)g'(x)h''(x) + g(x)h'(x)f''(x) + h(x)f'(x)g''(x) \\ &\quad -g(x)f'(x)h''(x) - f(x-"m)h'(x)g''(x) - h(x)g'(x)f''(x) \end{aligned} \tag{21.8}$$

THEOREM 21.4. *Suppose the functions $p_1(x), p_2(x), \ldots, p_n(x)$ are continuous (and differentiable) on the interval $\alpha < x < \beta$, and the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are solutions of*

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = 0 \quad . \tag{21.9}$$

Then if $W[y_1, y_2, \ldots, y_n](x) \neq 0$ at least one point in $\alpha < x < \beta$, then any solution of (21.9) can be expressed as a linear combination of the solutions $y_1(X), y_2(x), \ldots, y_n(x)$.

1. Solutions of the Non-homogeneous Problem

Consider a non-homogeneous n^{th} order linear differential equation of the form

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = g(x) \tag{21.10}$$

and suppose $y_1, y_2, \ldots, y_n$ is a set of n linearly independent solutions of the corresponding homogeneous problem. If $y_p(x)$ is any particular solution of (21.10), then the general solution of (21.10) can be written as

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \quad . \tag{21.11}$$

In an initial value problem the constants $c_1, \ldots, c_n$ are fixed uniquely by the set of initial conditions

$$\begin{aligned} y(x_o) &= y_o \\ y'(x_o) &= y'_o \\ &\vdots \\ y^{(n-1)}(x_o) &= y_o^{(n-1)} \quad . \end{aligned} \tag{21.12}$$

2. Linear Differential Equations with Constant Coefficients

Consider a differential equation of the form

$$a_n y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots a_1 y'(x) + a_0 y(x) = 0 \quad . \tag{21.13}$$

When $n = 2$ we know that a solution of this equation can be solved by making the ansatz

$$y(x) = e^{\lambda x} \tag{21.14}$$

plugging in and solving for λ. We can do the same thing for general n. Plugging (21.14) into (21.13) yields

$$0 = \left(a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0\right) e^{\lambda x} = 0 \tag{21.15}$$

and so a solution can be found for each root of the equation

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \quad . \tag{21.16}$$

Solutions of $\mathbf{n^{th}}$ order polynomial equations Theorem: Let $P(\lambda)$ be a polynomial with complex coefficients of degree n:

$$P(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \quad , \quad a_n, \ldots, a_0 \in \mathbb{C}. \tag{21.17}$$

If r is a root of $P(\lambda) = 0$, then $(\lambda - r)$ is a factor of $P(\lambda)$; that is to say, there exists a polynomial $Q(\lambda)$ such that

$$P(\lambda) = (\lambda - r)\, Q(\lambda) \quad . \tag{21.18}$$

Corollary: If $\{r_i, \ldots, r_p\}$ is the set of roots of a polynomial equation

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \tag{21.19}$$

then, there exists a unique set of positive integers $\{m_1, \ldots, m_p\}$ such that

$$P(\lambda) = (\lambda - r_1)^{m_1} (\lambda - r_2)^{m_2} \cdots (\lambda - r_p)^{m_p} . \tag{21.20}$$

Note that necessarily $m_1 + m_2 + \cdots + m_p = n$. The interger m_i corresponding to the i^{th} root r_i is called the *multiplicity* of the root r_i.

Theorem: If $P(\lambda)$ is a polynomial with real coefficients and $r = \alpha + i\beta \in \mathbb{C}$ is a root of $P(\lambda)$, then $\alpha^* = \alpha - i\beta$ is also a root of $P(\lambda)$.

Connection with Linear Homogeneous Differential Equations

Consider a differential equation of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0. \tag{21.21}$$

As noted above, if we make the substitution $y(x) = e^{\lambda x}$, we see that this differential equation for $y(x)$ is equivalent to the following algebraic equation for λ.

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0 = 0 \tag{21.22}$$

Let r be a root of this polynomial equation. There are four basic cases.

(i) r is a distinct real root.
In this case, we have a distinct solution of the form

$$y(x) = e^{rx} \quad . \tag{21.23}$$

(ii) $r = \alpha \pm i\beta$ are distinct complex roots.
In this case we have two distinct solutions

$$y_1(x) = e^{\alpha x}\cos(\beta x) \quad , \quad y_2(x) = e^{\alpha x}\sin(\beta x) \quad . \tag{21.24}$$

(iii) r is a real root with multiplicity k.
In this case, one can show that the functions

$$\begin{array}{rcl} y_1(x) & = & e^{rx} \\ y_2(x) & = & xe^{rx} \\ \vdots & & \\ y_k(x) & = & x^{k-1}e^{rx} \end{array} \tag{21.25}$$

comprise a set of k linearly independent solutions of (21.13).

(iv) $r = \alpha \pm i\beta$ are complex roots each with multiplicity k.
In this case, one can show that the functions

$$\begin{array}{rcl} y_1(x) & = & e^{\alpha x}\cos(\beta x) \\ y_2(x) & = & xe^{\alpha x}\cos(\beta x) \\ \vdots & & \\ y_k(x) & = & x^{k-1}e^{\alpha x}\cos(\beta x) \\ y_{k+1}(x) & = & e^{\alpha x}\sin(\beta x) \\ y_{k+2}(x) & = & xe^{\alpha x}\sin(\beta x) \\ \vdots & & \\ y_{2k}(x) & = & x^{k-1}e^{\alpha x}\sin(\beta x) \end{array} \tag{21.26}$$

form a set of $2k$ linearly independent solutions.

As a polynomial of the form (21.16) always factors as

$$(x-r_1)^{k_1}\cdots(x-r_i)^{k_i}\left(x-\alpha_1-i\beta_1\right)^{l_1}\left(x-\alpha_1+i\beta_1\right)^{l_1} \cdots\left(x-\alpha_j-i\beta_j\right)^{l_j}\left(x-\alpha_j+i\beta_j\right)^{l_j} \tag{21.27}$$

with

$$n=\sum k_i+\sum 2l_j \tag{21.28}$$

in this manner one can always write down n linearly independent solutions of (21.13).

Example 21.5. Find the general solution of

$$y'''-y''-y'+y=0 \quad . \tag{21.29}$$

: The characteristic equation for this differential equation is

$$\lambda^3-\lambda^2-\lambda+1=0 \tag{21.30}$$

or

$$(\lambda-1)^2(\lambda+1)=0. \tag{21.31}$$

We thus have a double root at $\lambda=1$ and a single root at $\lambda=-1$. The general solution is thus

$$y(x)=c_1e^x+c_2xe^x+c_3e^{-x} \quad . \tag{21.32}$$

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Example 21.6. Find the general solution of

$$\frac{d^6y}{dx^6}+y=0 \quad . \tag{21.33}$$

: In this case the characteristic equation is

$$\lambda^6+1=0 \quad . \tag{21.34}$$

Thus, λ must be one of the roots of

$$\lambda^6=-1=e^{i\pi} \tag{21.35}$$

Thus,

$$\begin{aligned}\lambda &= \pm e^{\pm\frac{i\pi}{6}}, e^{\pm\frac{i\pi}{2}} \\ &= \pm\left(\cos\left(\tfrac{\pi}{6}\right)\pm i\sin\left(\tfrac{\pi}{6}\right)\right), \pm\left(\cos\left(\tfrac{\pi}{2}\right)\pm i\sin\left(\tfrac{\pi}{2}\right)\right) \\ &= \tfrac{\sqrt{3}}{2}\pm\tfrac{i}{2}, -\tfrac{\sqrt{3}}{2}\pm\tfrac{i}{2}, \pm i \quad .\end{aligned} \tag{21.36}$$

So we have 6 distinct roots and

$$y(x)=c_1e^{\frac{\sqrt{3}}{2}x}\cos\left(\frac{x}{2}\right)+c_2e^{\frac{\sqrt{3}}{2}x}\sin\left(\frac{x}{2}\right)+c_3e^{-\frac{\sqrt{3}}{2}x}\cos\left(\frac{x}{2}\right)+c_4e^{-\frac{\sqrt{3}}{2}x}\sin\left(\frac{x}{2}\right)+c_5\cos(x)+c_6\sin(x) \tag{21.37}$$

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