LECTURE 20

Variation of Parameters

Consider the differential equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{20.1}$$

Suppose $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous problem corresponding to (20.1); i.e., y_1 and y_2 satisfy

$$y'' + p(x)y' + q(x)y = 0 \tag{20.2}$$

and

$$W[y_1, y_2] \neq 0 \quad . \tag{20.3}$$

We seek to determine two functions $u_1(x)$ and $u_2(x)$ such that

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{20.4}$$

is a solution of (20.1). To determine the two functions u_1 and u_2 uniquely we need to impose two (independent) conditions. First, we shall require (20.4) to be a solution of (20.1); and second, we shall require

$$u_1'y_1 + u_2'y_2 = 0 \quad . \tag{20.5}$$

(This latter condition is imposed not only because we need a second equation, but also to make the calculation a lot easier.)

Differentiating (20.4) yields

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \tag{20.6}$$

which because of (20.5) becomes

$$y_p' = u_1y_1' + u_2y_2' \quad . \tag{20.7}$$

Differentiating again yields

$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \quad . \tag{20.8}$$

We now plug (20.4), (20.7), and (20.8) into the original differential equation (20.1).

$$\begin{aligned} g(x) &= (u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p(x)\left(u_1y_1' + u_2y_2'\right) + q(x)\left(u_1y_1 + u_2y_2\right) \\ &= u_1'y_1' + u_2'y_2' + u_1\left(y_1'' + p(x)y_1' + q(x)y_1\right) + u_2\left(y_2'' + p(x)y_2' + q(x)y_2\right) \end{aligned} \tag{20.9}$$

The last two terms vanish since y_1 and y_2 are solutions of (20.2). We thus have

$$u_1'y_1 + u_2'y_2 = 0 \tag{20.10}$$

$$u_1'y_1' + u_2'y_2' = g \tag{20.11}$$

We now can now solve this pair of equations for u_1 and u_2. The result is

$$\begin{aligned} u_1' &= \frac{-y_2g}{y_1y_2' - y_1'y_2} = \frac{-y_2g}{W[y_1,y_2]} \\ u_2' &= \frac{y_1g}{y_1y_2' - y_1'y_2} = \frac{y_1g}{W[y_1,y_2]} \quad . \end{aligned} \tag{20.12}$$

(Note that division by $W(y_1, y_2)$ causes no problems since y_1 and y_2 were chosen such that $W(y_1, y_2) \neq 0$.) Hence

$$\begin{array}{rcl} u_1(x) & = & \int^x \frac{-y_2(t)g(t)}{W[y_1,y_2](t)} dt \\ u_2(x) & = & \int^x \frac{y_1(t)g(t)}{W[y_1,y_2](t)} dx' \end{array} \tag{20.13}$$

and so

$$\boxed{y_p(x) = -y_1(x) \int^x \frac{y_2(t)g(t)}{W[y_1,y_2](t)} dt + y_2(x) \int^x \frac{y_1(t)g(t)}{W[y_1,y_2](t)} dt} \tag{20.14}$$

is a particular solution of (20.1).

EXAMPLE 20.1. Find the general solution of

$$y'' - y' - 2y = 2e^{-x} \tag{20.15}$$

using the method of Variation of Parameters.

Well, the corresponding homogeneous problem is

$$y'' - y' - 2y = 0 \quad . \tag{20.16}$$

This is a second order linear equation with constant coefficients whose characteristic equation is

$$\lambda^2 - \lambda - 2 = 0 \quad . \tag{20.17}$$

The characteristic equation has two distinct real roots

$$\lambda = -1, 2 \tag{20.18}$$

and so the functions

$$\begin{array}{rcl} y_1(x) & = & e^{-x} \\ y_2(x) & = & e^{2x} \end{array} \tag{20.19}$$

form a fundamental set of solutions to (20.16).

To find a particular solution to (20.15) we employ the formula (20.14). Now

$$g(x) = 2e^{-x} \tag{20.20}$$

and

$$W[y_1, y_2](x) = \left(e^{-x}\right)\left(2e^{2x}\right) - \left(-e^{-x}\right)\left(e^{2x}\right) = 3e^x \quad , \tag{20.21}$$

so

$$\begin{array}{rcl} y_p(x) & = & -y_1(x) \int^x \frac{y_2(t)g(t)}{W[y_1,y_2](t)} dt \quad + \quad y_2(x) \int^x \frac{y_1(t)g(t)}{W[y_1,y_2](t)} dt \\ & = & -e^{-x} \int^x \frac{e^{2t}\left(2e^{-t}\right)}{3e^t} dt \quad + \quad e^{2x} \int^x \frac{e^{-t}\left(2e^{-t}\right)}{3e^t} dt \\ & = & -e^{-x} \int^x \frac{2}{3} dt \quad + \quad e^{2x} \int^x \frac{2}{3} e^{-3t} dt \\ & = & -\frac{2}{3} x e^{-x} - \frac{2}{9} e^{-x} \end{array} \tag{20.22}$$

The general solution of (20.15) is thus

$$\begin{array}{rcl} y(x) & = & y_p(x) + c_1 y_1(x) + c_2(x) \\ & = & -\frac{2}{3} x e^{-x} + \left(c_1 - \frac{2}{9}\right) e^{-x} + c_2 e^{2x} \\ & = & -\frac{2}{3} x e^{-x} + C_1 e^{-x} + C_2 e^{2x} \end{array} \tag{20.23}$$

where we have absorbed the $-\frac{2}{9}$ in the second line into the arbitrary parameter C_1.